Chapter 3: Motion in two or three dimensions

Position as a vector

The position vector is a vector that points from the origin of the *chosen* coordinate system to the point that a particle is located.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

The displacement vector points between two position vectors.

$$\begin{aligned}\Delta\vec{r} &= \vec{r} - \vec{r}_0 \\ &= (x - x_0)\,\hat{i} + (y - y_0)\,\hat{j} + (z - z_0)\,\hat{k}\end{aligned}$$

Velocity as a vector

The average velocity vector is the displacement vector between two points divided by the change in time,

$$\begin{aligned} \vec{v}_{\mathbf{avg}} &= \frac{\vec{r} - \vec{r}_0}{t - t_0} \\ &= \frac{\Delta \vec{r}}{\Delta t} \end{aligned}$$

It follows that the instantaneous velocity vector is given by

$$\begin{aligned} \vec{v} &= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \\ &= \frac{d\vec{r}}{dt} \end{aligned}$$

We may write the instantaneous velocity in terms of Cartesian coordinates,

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) \\ &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \end{aligned}$$

Components, magnitude, & direction of velocity

From the previous equation, we see that the velocity vector can be separated into the three Cartesian coordinate components

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$

The magnitude of the velocity vector can be written in terms of the velocity components,

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

For two-dimensional motion in the *xy*-plane, the *z*-component is zero, and the equation reduces to

$$v = \sqrt{v_x^2 + v_y^2}$$

For two-dimensional motion in the *xy*-plane, the angle measured counter-clockwise from the +*x*-axis is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \Big| \quad \theta = \pi + \tan^{-1}\left(\frac{v_y}{v_x}\right) \text{ for } \frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

Acceleration as a vector

The average acceleration vector is the change in the velocity vector divided by the change in time,

$$\vec{a}_{\text{avg}} = \frac{\Delta\vec{v}}{\Delta t}$$

It follows that the instantaneous acceleration vector is given by

$$\begin{aligned}\vec{a} &= \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t}\\ &= \frac{d\vec{v}}{dt}\end{aligned}$$

In a similar manner to the velocity vector, we may write the instantaneous acceleration in terms of Cartesian coordinates,

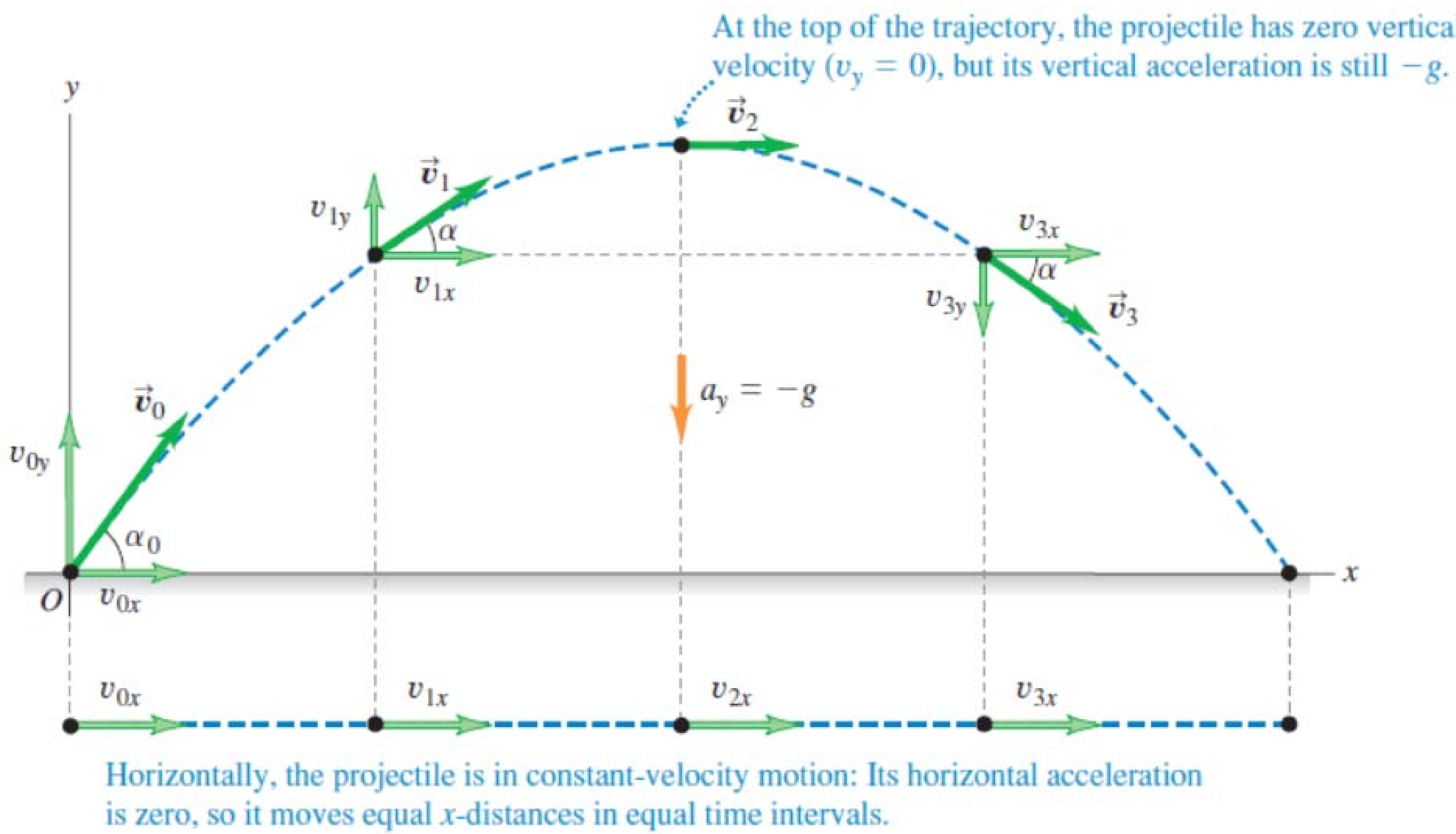
$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt}\\ &= \frac{d}{dt}\left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)\\ &= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}\\ &= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}\end{aligned}$$

Components of the acceleration

We can separate the acceleration vector into the three cartesian coordinate components

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

Projectile motion



Projectile motion equations

(air resistance is neglected)

For projectile motion, we have no acceleration in the horizontal direction during the time-of-flight, and we have a downward acceleration due to gravity ($g = 9.8\ \mathrm{m/s^2}$).

$$
\begin{aligned}
x &= x_0 + v_{0x} t \\
y &= y_0 + v_{0y} t - \frac{1}{2} g t^2
\end{aligned}
$$

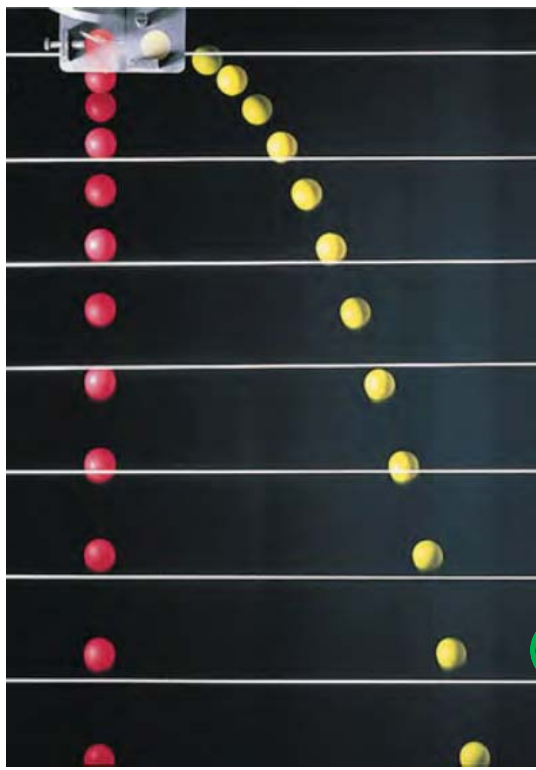
If given the magnitude of velocity and angle above the horizon, then the initial velocity components are

$$
\begin{aligned}
v_{0x} &= v_0 \cos\theta \\
v_{0y} &= v_0 \sin\theta
\end{aligned}
$$

Horizontal and vertical motions are <u>independent</u> of each other.

At t=0, two identical balls are released.

One drops straight down and the other is launched horizontally.



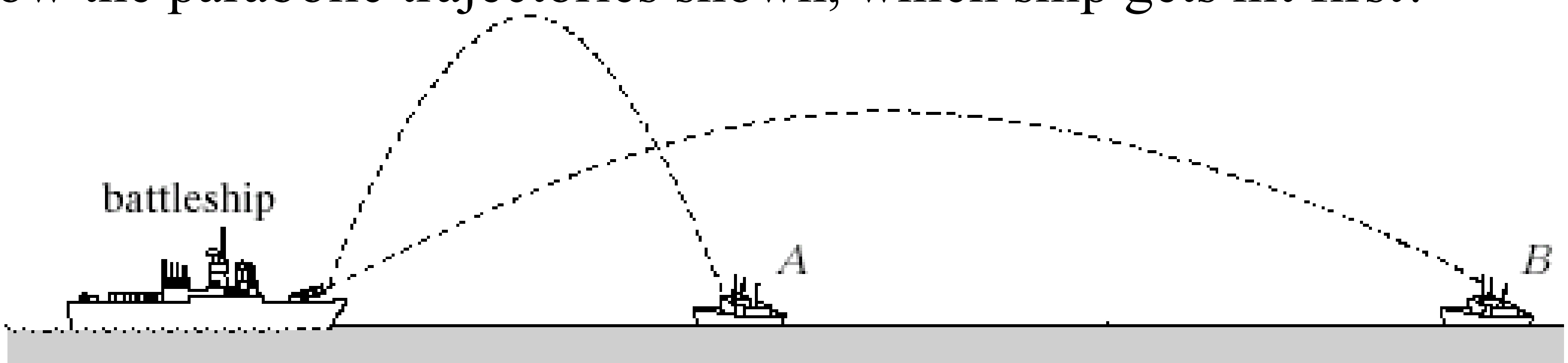
Which ball will hit the ground first?

A. Dropped ball

B. Launched ball

C. Both hit at the same time

A battleship simultaneously fires two shells at enemy ships. If the shells follow the parabolic trajectories shown, which ship gets hit first?

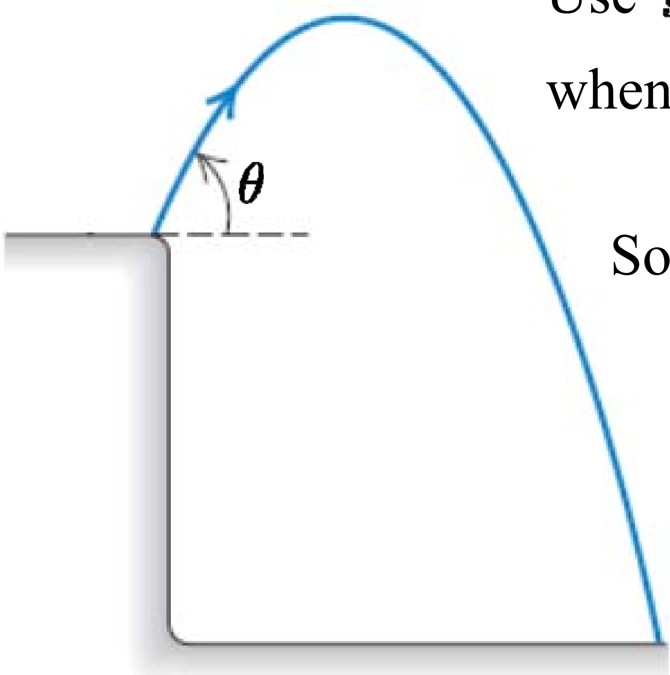


1. *A*
2. both at the same time
3. *B*
4. need more information

The acceleration vector of a particle in projectile motion

A. points along the path of the particle.
B. is directed horizontally.
C. vanishes at the particle’s highest point.
D. is directed down at all times.
E. is zero.

Finding the **range** if we know the magnitude & direction of the velocity and the height



Use $y = y_0 + v_0 \sin\theta\, t - \frac{1}{2} g t^2$ where $y = 0$ when the projectile lands and begins at a height, $y_0 = h$.

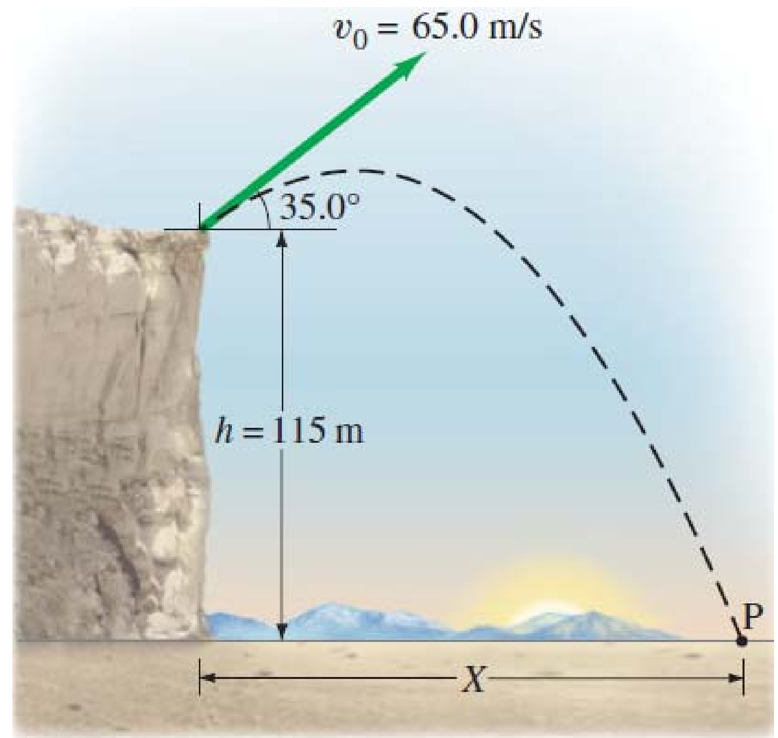
Solve the equation $0 = h + v_0 \sin\theta\, t - \frac{1}{2} g t^2$

$$\boxed{t = \frac{v_0 \sin\theta}{g} + \sqrt{\frac{v_0^2 \sin^2\theta}{g^2} + \frac{2h}{g}}}$$

Plug time into $x = x_0 + v_0 \cos\theta\, t$

Assuming $x_0 = 0$, we get $$\boxed{x = \frac{v_0^2 \sin\theta \cos\theta}{g} + \sqrt{\frac{v_0^4 \sin^2\theta \cos^2\theta}{g^2} + \frac{2h v_0^2 \cos^2\theta}{g}}}$$

A projectile is shot from the edge of a cliff 115 m above ground level with an initial speed of 65.0 m/s at an angle of 35.0° with the horizontal.



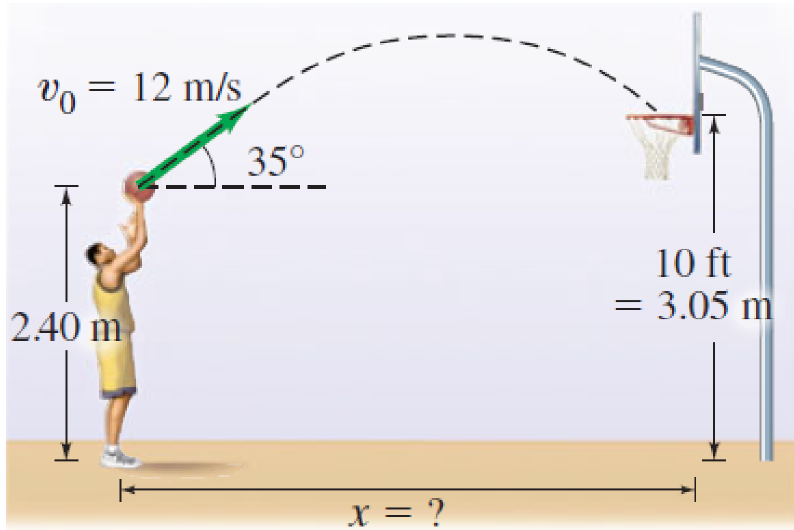
(a) Determine the time taken by the projectile to hit point P at ground level.

(b) Determine the distance X of point P from the base of the vertical cliff.

(c) At the instant just before the projectile hits point P, find the horizontal and the vertical components of its velocity.

(d) At the instant just before the projectile hits point P, find the magnitude of the velocity.

(e) At the instant just before the projectile hits point P, find the angle made by the velocity vector with the horizontal.



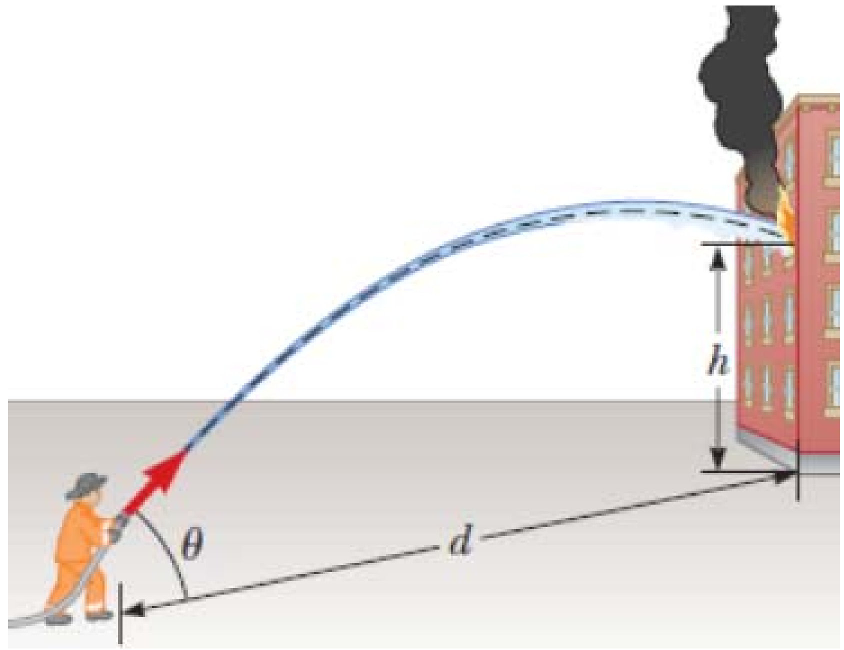
A basketball is shot by an HPU Shark from an initial height of 2.40 m with an initial speed of $v_0 = 12$ m/s directed at an angle $\theta_0 = 35^o$ above the horizontal. The basket is at a fixed height of 3.05 m.

(a) Determine the time taken for the ball to enter the hoop.

(b) Determine the distance between the player and the basket.

(c) Determine the magnitude of the basketball's velocity just as it enters the basket.

(d) Determine what angle to the horizontal the ball enters the basket.

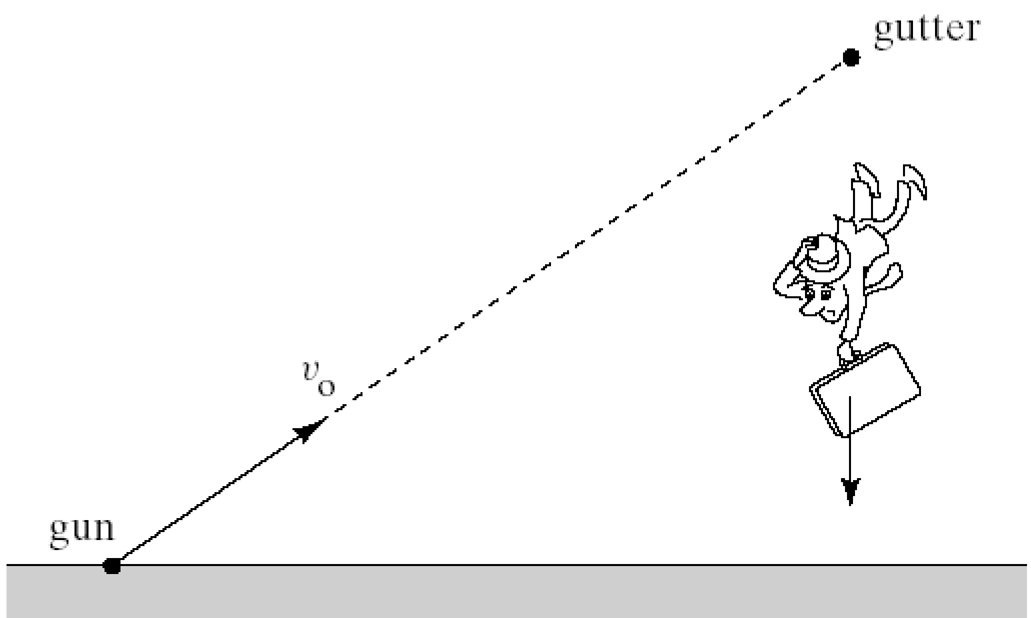


A firefighter, a distance d from a burning building, directs a stream of water from a fire hose at angle θ above the horizontal. If the initial speed of the stream is v_0, at what height h does the water strike the building?

A cannon is on a city wall 20 m above the ground facing a flat horizontal field. It shoots a projectile at an angle of 30º above the horizon and lands 120 m from the edge of the city wall. What is the magnitude of the initial velocity?

Consider the situation depicted here. A gun is accurately aimed at a dangerous criminal hanging from the gutter of a building. The target is well within the gun's range, but the instant the gun is fired and the bullet moves with a speed v_o, the criminal lets go and drops to the ground. What happens?

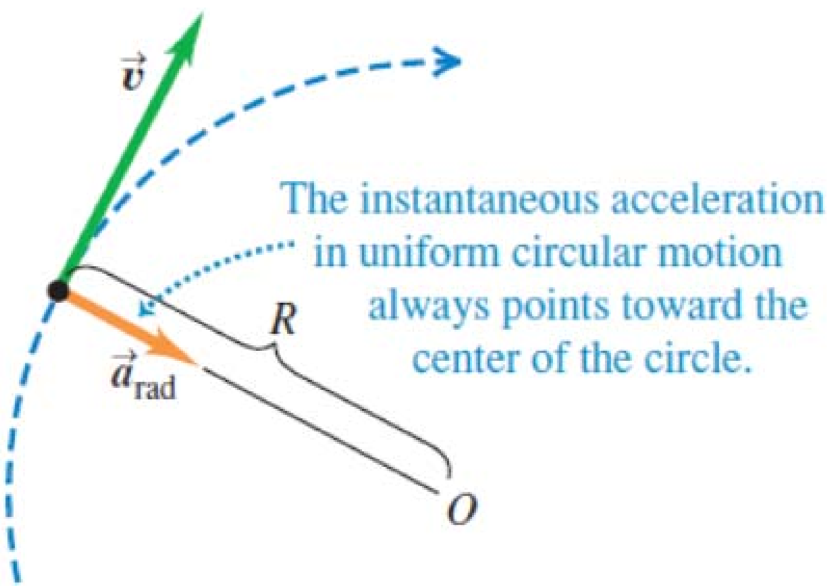
The bullet …



1. hits the criminal
2. misses the criminal.

Circular motion

For uniform circular motion the ACCELERATION is directed toward the center of the circle at all times.



The magnitude of the radial acceleration is given by

$$\boxed{a_{\text{rad}} = \frac{v^2}{r}}$$

Acceleration to remain in const. circular motion

The acceleration is constant and directed radially inward when an object is in uniform circular motion.

$$a_r = \text{const.}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{\Delta}{\Delta t}\left(\frac{\Delta s}{\Delta t}\right)$$

For one revolution, the arc length traveled is $\Delta s = 2\pi r$ over a time period T.

Therefore, $\dfrac{\Delta s}{\Delta t} = \dfrac{2\pi r}{T}$.

The change in the this quantity is related to the change in direction of the velocity, which is 2π over the time period T.

Therefore, $\dfrac{\Delta}{\Delta t}\left(\dfrac{\Delta s}{\Delta t}\right) = \dfrac{4\pi^2 r}{T^2}$.

For constant motion, $v = \dfrac{2\pi r}{T}$ and therefore, $a_r = \dfrac{v^2}{r} = \dfrac{4\pi^2 r}{T^2}$.

Nonuniform circular motion

In nonuniform circular motion, $a_r = \dfrac{v^2}{r}$ still gives the radial component of acceleration which is always perpendicular to the instantaneous velocity and directed toward the center of the circle.

The speed varies and so $a_r \neq \mathrm{const.}$

The change in speed means that there is a tangential component to the acceleration along the direction of motion.

The magnitude of the tangential acceleration is $a_{\tan} = \dfrac{d\left|\vec{v}\right|}{dt}$, while the magnitude of the radial acceleration remains $a_r = \left|\dfrac{d\vec{v}}{dt}\right|$.

When a ball on the end of a string swings in a vertical circle with uniform circular motion, the ball is accelerating because

A. the speed is changing.
B. the direction is changing.
C. the speed and the direction are changing.
D. the ball is not accelerating.

When a ball on the end of a string is swung in a vertical circle with uniform motion:

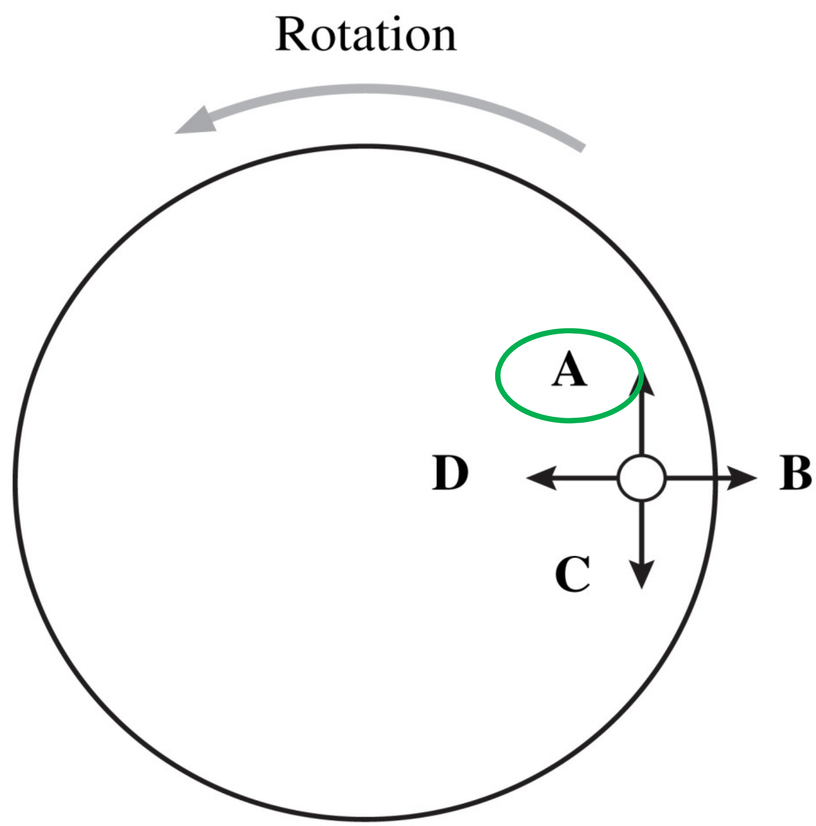
What is the direction of the acceleration of the ball?

A. Tangent to the circle, in the direction of the ball's motion

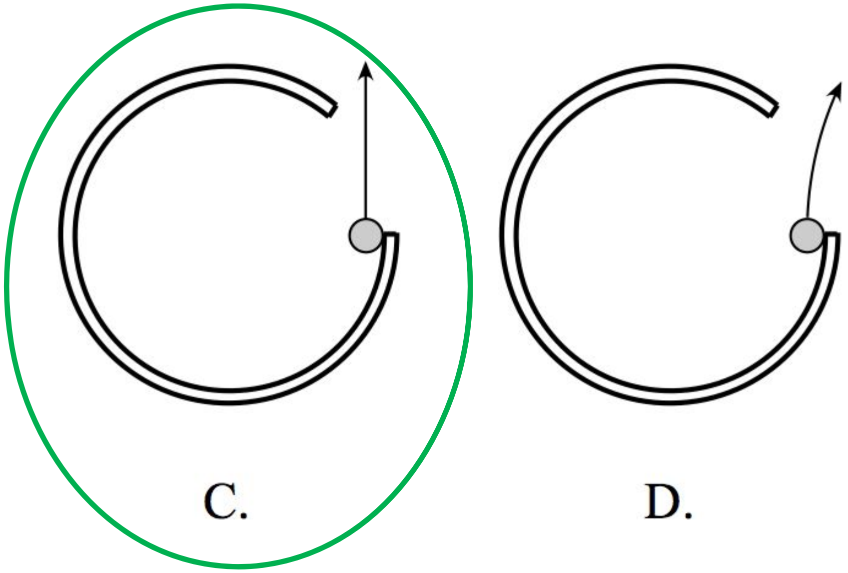
B. Toward the center of the circle

A coin sits on a rotating turntable.

At the time shown in the figure, which arrow gives the direction of the coin's velocity?

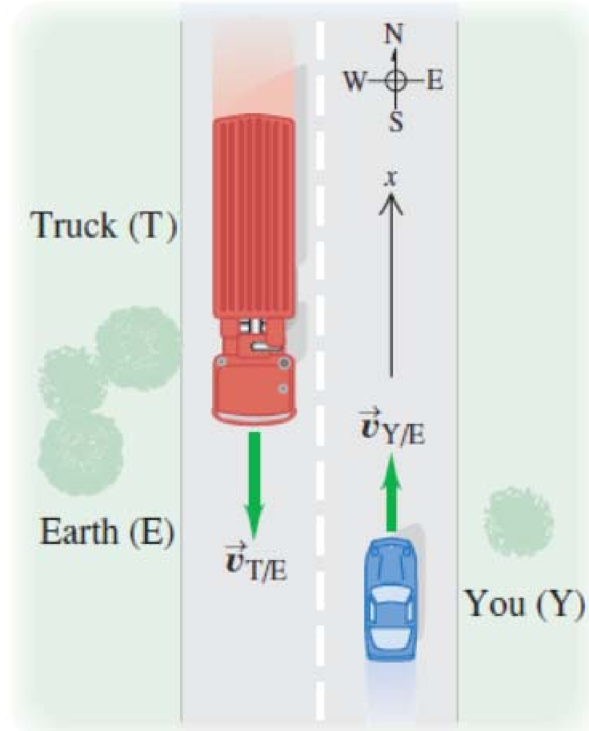


When the ball reaches the break in the circle, which path will it follow?



A. B. C. D.

Relative velocity



$v_T = v_{T/E} =$ velocity of the truck with respect to a nearby point on Earth.

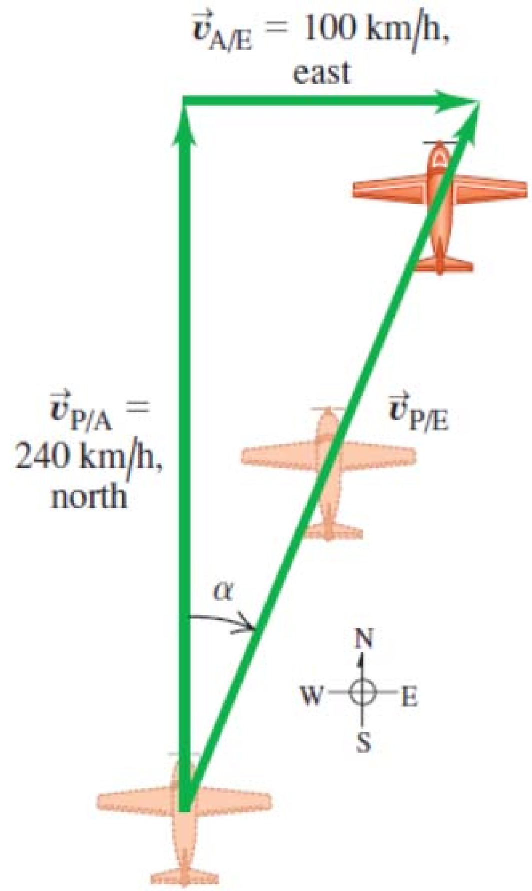
$v_Y = v_{Y/E} =$ your velocity with respect to a nearby point on Earth.

$$v_{Y\ (\text{with respect to Earth})} = v_{Y\ (\text{with respect to truck})} + v_{T\ (\text{with respect to Earth})}$$

Therefore, $v_{Y/E} = v_{Y/T} + v_{T/E}$ **or** $\boxed{v_{Y/T} = v_{Y/E} - v_{T/E}}$

Relative velocity in 2-dimensions

An airplane's compass indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 240 km/h. If there is a 100 km/h wind from west to east, what is the magnitude of the airplane's velocity relative to the earth?



$$v_{P/E} = \sqrt{v_{P/A}^2 + v_{A/E}^2} = 260\,\mathrm{km/h}$$

A stunt pilot, flying a constant 208 km/h horizontally in a ultralight helicopter, wants to drop a package into an open car which is traveling 156 km/h on a level highway $h = 82.0$ m below. At what angle below the horizontal should the car be in her sights when the packet is released?