Chapter 8: momentum, impulse, and collisions

Definition of momentum

The linear momentum of an object can be set in terms of Newton's second law, where

$$\begin{aligned} \sum \vec{F} &= m\vec{a} \\ &= m\frac{d\vec{v}}{dt} \end{aligned}$$

When the mass of an object is constant, the mass may be moved to the right of the time derivative,

$$\begin{aligned} m\frac{d\vec{v}}{dt} &= \frac{d}{dt}\left(m\vec{v}\right) \\ &= \frac{d\vec{p}}{dt} \end{aligned}$$

where $\vec{p} = m\vec{v}$ is the linear momentum.

Impulse-momentum theorem

Consider a particle acted on by a constant net force $\sum \vec{F}$ during a time interval Δt. The **impulse** of the net force, denoted by $\vec{J}$ is defined to be the product of the net force and the time interval

$$\vec{J} = \sum \vec{F} \Delta t$$

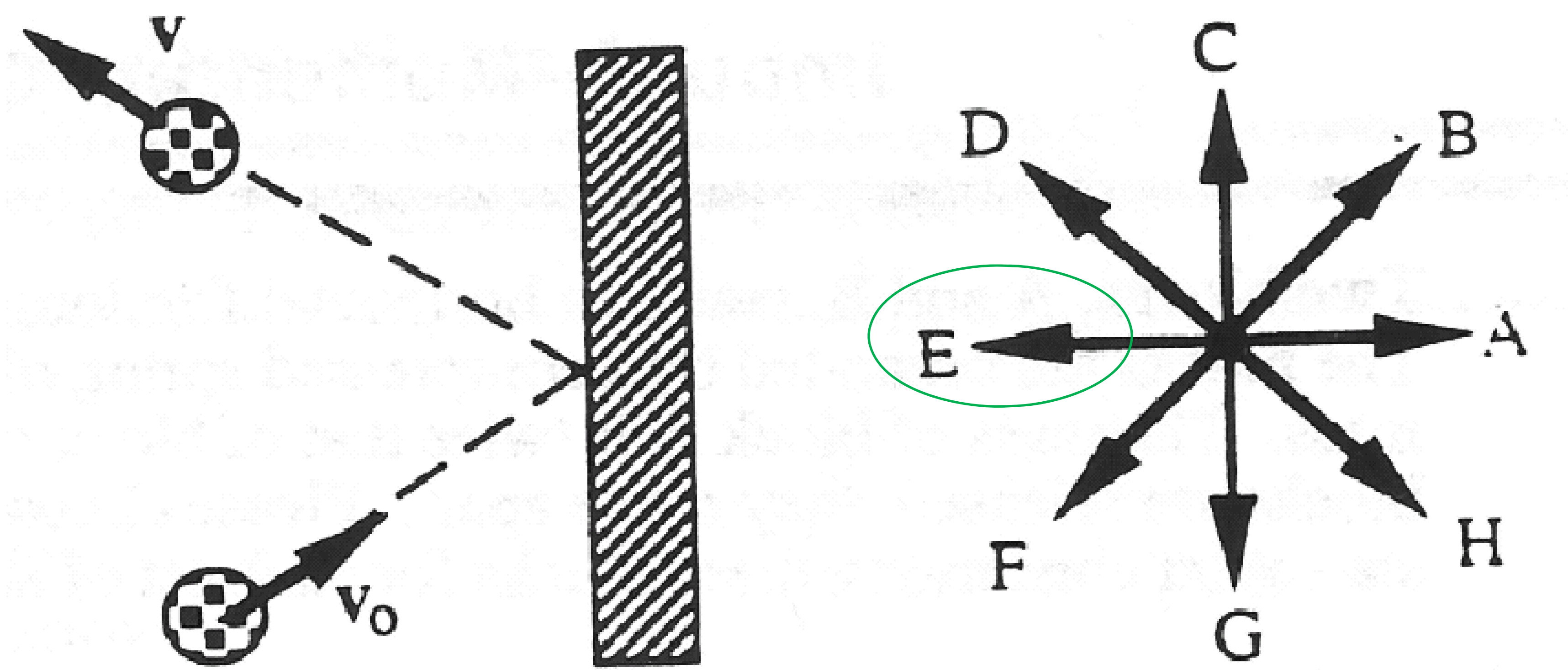
The infinitesimal change in the impulse is given by the infinitesimal change in time, which holds true for non-constant forces,

$$\begin{aligned} \int_0^{\vec{J}} d\vec{J}' &= \int_0^t \left(\frac{d\vec{p}'}{dt'} \right) dt' \\ &= \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p}' \\ &= \vec{p}_f - \vec{p}_i \end{aligned}$$

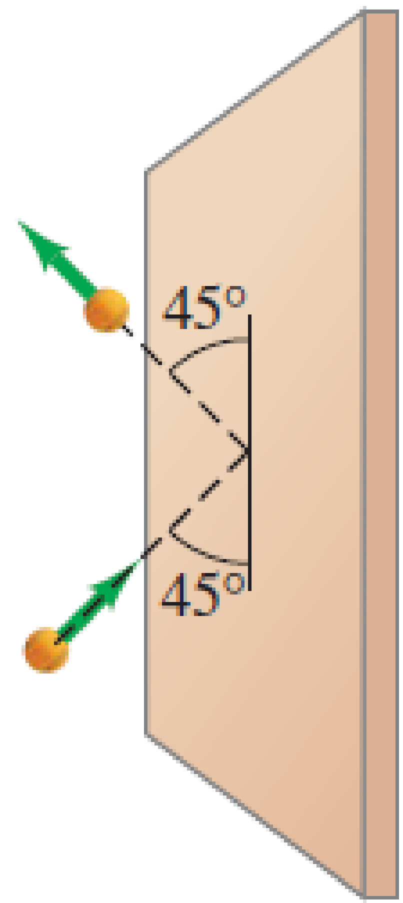
Therefore, we can relate impulse and momentum via

$$\vec{J} = \Delta \vec{p}$$

A ball hits a wall as shown in the figure. In which direction is the impulse of the wall on the ball? (Choose from A-G)



A tennis ball of mass $m = 0.060$ kg and speed $v = 28.0$ m/s strikes a wall at a 45° angle and rebounds with the same speed at 45°. What is the impulse (magnitude and direction) given to the ball?



If the ball made contact with the wall for a period of $\Delta t = 20$ ms, then what is the average force that the wall exerts on the ball?

We are usually interested in the <u>CHANGES</u> in momentum:

A bug collides with the windshield of a car traveling down the highway. Which experiences a greater force?

Which experiences a greater ***change*** in momentum?

1. it depends on the initial velocity of the bug
2. the car
3. the bug
4. both experience the same change
5. none of the above

Suppose a ping-pong ball and a bowling ball are rolling toward you. Both have the same momentum, and you exert the same force to stop each. How do the time intervals to stop them compare?

1. It takes less time to stop the ping-pong ball.
2. Both take the same time.
3. It takes more time to stop the ping-pong ball.

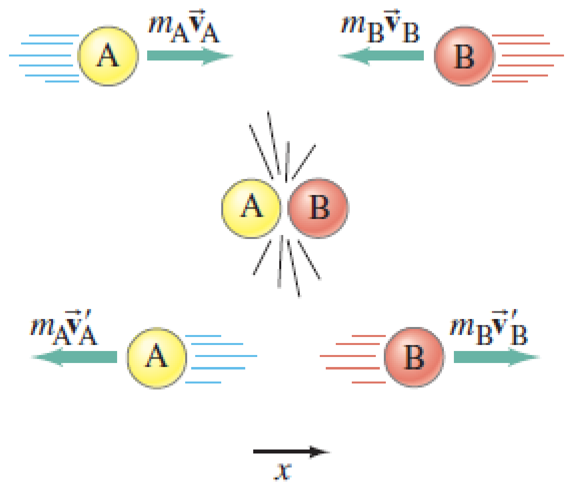
Conservation of Momentum

If no net external force acts on a system, the total momentum of the system is a conserved quantity.

$$\Delta \vec{p} = 0$$

or

$$\vec{p}_i = \vec{p}_f$$

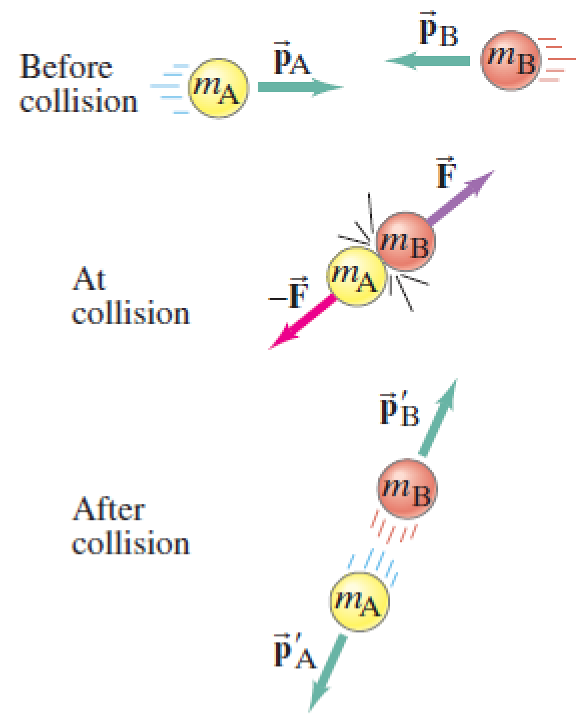


This can be rewritten in terms of the mass and the velocity of the objects colliding.

$$m_A\vec{v}_{Ai} + m_B\vec{v}_{Bi} = m_A\vec{v}_{Af} + m_B\vec{v}_{Bf}$$

Conservation of Momentum & Newton's 3rd law

During the collision, the objects exert forces on each other



$$\vec{F}_{\text{B on A}} = \frac{d\vec{p}_B}{dt}$$

There must be an equal and opposite force in the same time interval,

$$\vec{F}_{\text{A on B}} = \frac{d\vec{p}_A}{dt}$$

where $\vec{F}_{\text{B on A}} = -\vec{F}_{\text{A on B}}$

Adding the two forces together gives $\dfrac{d\vec{p}_A}{dt} + \dfrac{d\vec{p}_B}{dt} = 0$

Therefore, $\displaystyle\int_{\vec{p}_{Ai}}^{\vec{p}_{Af}} d\vec{p}_A{}' + \int_{\vec{p}_{Bi}}^{\vec{p}_{Bf}} d\vec{p}_B{}' = 0$ or $\vec{p}_{Ai} + \vec{p}_{Bi} = \vec{p}_{Af} + \vec{p}_{Bf}$

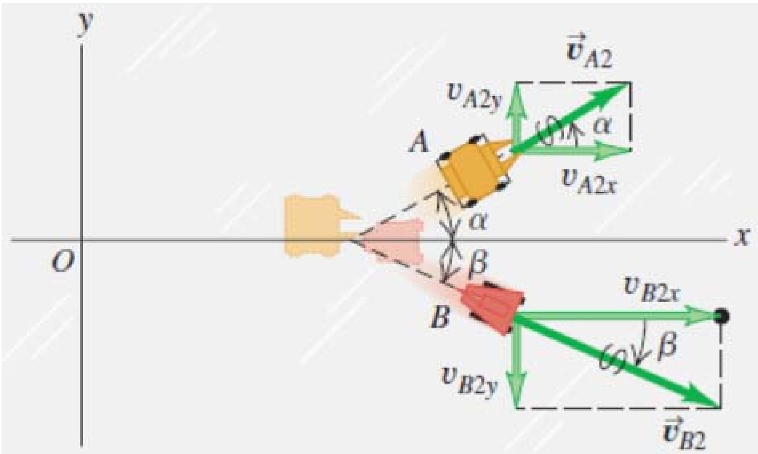
Conservation of momentum components

Conservation of momentum is written as the vector equation

$$\vec{p}_i = \vec{p}_f$$

Thus, we can break this equation into separate equations for the Cartesian components in the plane of interaction

$$\begin{cases} p_{xi} = p_{xf} \\ p_{yi} = p_{yf} \end{cases}$$



Suppose you are on a cart, initially at rest on a track with very little friction. You throw balls at a partition that is rigidly mounted on the cart. If the balls bounce straight back as shown in the figure, is the cart put in motion?

1. Yes, it moves to the right.
2. Yes, it moves to the left.
3. No, it remains in place.
4. It depends on the mass of the ball and the cart, and the velocity of the throw.

Elastic & Inelastic collisions

In an **elastic collision**, the kinetic energy is also conserved during the collision.

$$\Delta\vec{p} = 0 \quad \text{and} \quad \Delta K = 0$$

In an **inelastic collision**, the kinetic energy is NOT conserved during the collision.

$$\Delta\vec{p} = 0 \quad \text{but} \quad \Delta K \neq 0$$

The kinetic energy that is lost is changed into other forms of energy,
often thermal energy, so that the total energy (as always) is conserved.

ELASTIC collisions in 1 dimension

(elastic collisions are the only time mechanical energy is conserved)

We can write down the conservation of momentum equation

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

We can write down the conservation of kinetic energy equation

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Rearranging the conservation of kinetic energy equation and the conservation of momentum equation via algebraic methods gives

$$m_1 v_{1i}^2 - m_1 v_{1f}^2 = m_2 v_{2f}^2 - m_2 v_{2i}^2$$

and $$m_1 v_{1i} - m_1 v_{1f} = m_2 v_{2f} - m_2 v_{2i}$$

Simplifying the conservation of kinetic energy equation and dividing by the conservation of momentum equation gives

$$\frac{m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f})}{m_1 (v_{1i} - v_{1f})} = \frac{m_2 (v_{2f} - v_{2i})(v_{2f} + v_{2i})}{m_2 (v_{2f} - v_{2i})}$$

Thus, we arrive at a simplified equation for the elastic collision from the conservation of energy,

$$\boxed{v_{1f} + v_{1i} = v_{2f} + v_{2i}}$$

Explosions: inelastic collisions in reverse

An object initially at rest with a mass of $m_{tot} = 6.5$ kg explodes into two fragments, the first has a mass of $m_A = 4.0$ kg, and the other has a mass of $m_B = 2.5$ kg. The explosion perfectly transfers the energy $Q = 150$ J into kinetic energy of the fragments.

Calculate the kinetic energy of the object with mass m_A after the explosion.

Calculate the kinetic energy of the object with mass m_B after the explosion.

Kinetic energy and momentum

We first learned to write down the expression for the kinetic energy of a moving object as

$$K = \frac{1}{2}mv^2$$

We now have an expression for the momentum of an object with mass,

$$p = mv$$

Solving for the velocity in terms of momentum gives,

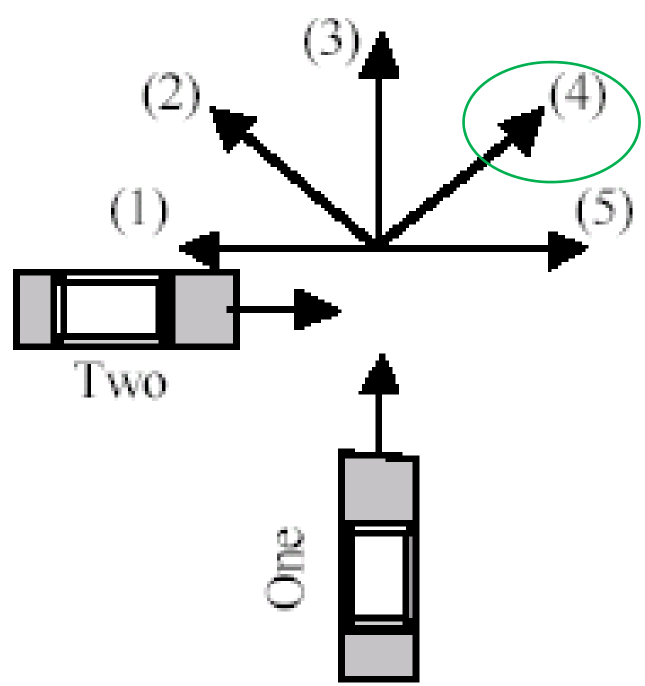
$$v = \frac{p}{m}$$

Substituting the velocity into the kinetic energy equation results in a kinetic energy expression that depends on the momentum,

$$\begin{aligned} K &= \frac{1}{2}m\left(\frac{p}{m}\right)^2 \\ &= \frac{p^2}{2m} \end{aligned}$$

Collisions in two dimensions

Car One is traveling due north and Car Two is traveling due east. After the collision shown, Car One rebounds due south. Which of the numbered arrows is the only one that can represent the final direction of Car Two?

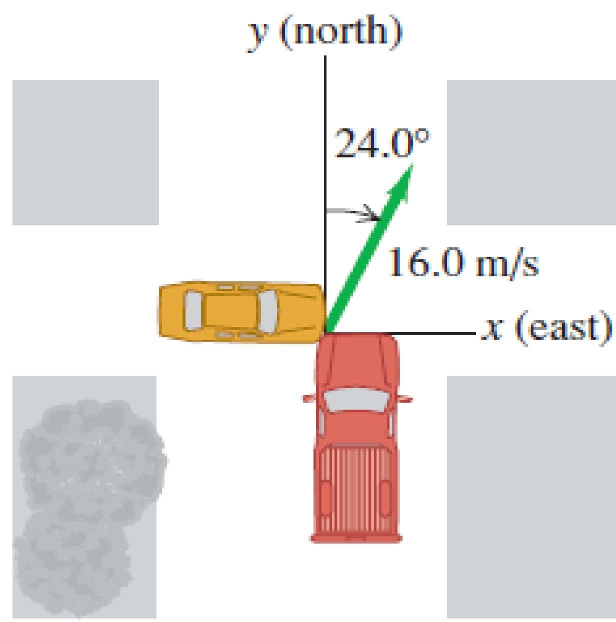


A car with a mass of $m_C = 950$ kg travels east and collides with a truck with a mass of $m_T = 1900$ kg that is traveling north that did not stop at red light. The two vehicles stick together as a result of the collision, and the wreckage slides at $v_f = 16.0$ m/s in the direction $\theta = 24.0°$ east of north.

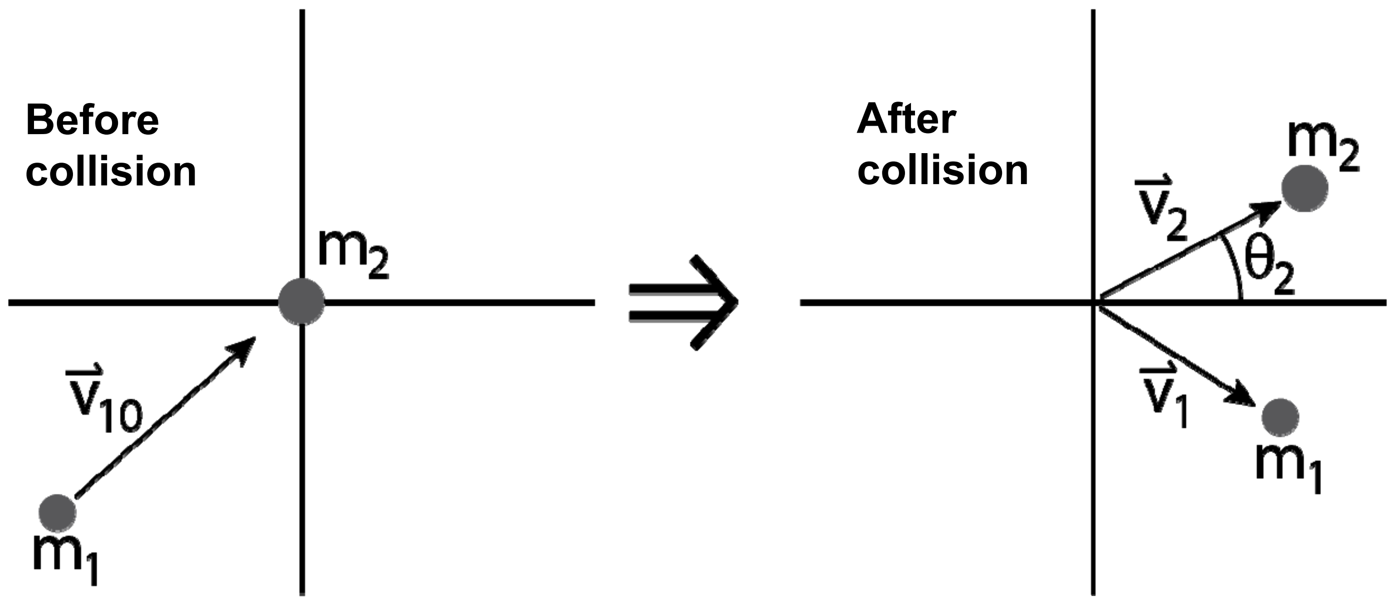
Determine the initial speed of the car, v_{Ci}.

Determine the initial speed of the truck, v_{Ti}.

Determine the change in kinetic energy before and after the collision, ΔK.



An object with mass $m_1 = 1.00$ kg has a velocity of $|\vec{v}_{1i}| = 2.00$ m/s $\hat{\imath}$ + 2.00 m/s $\hat{\jmath}$. It collides with a second object of mass $m_2 = 2.00$kg that is initially at rest. After the collision we measure the first block to have a magnitude of velocity $|\vec{v}_{1f}|= 1.50$ m/s at an angle of $\theta_1 = 330^o$ as measured counter-clockwise from the x-axis.



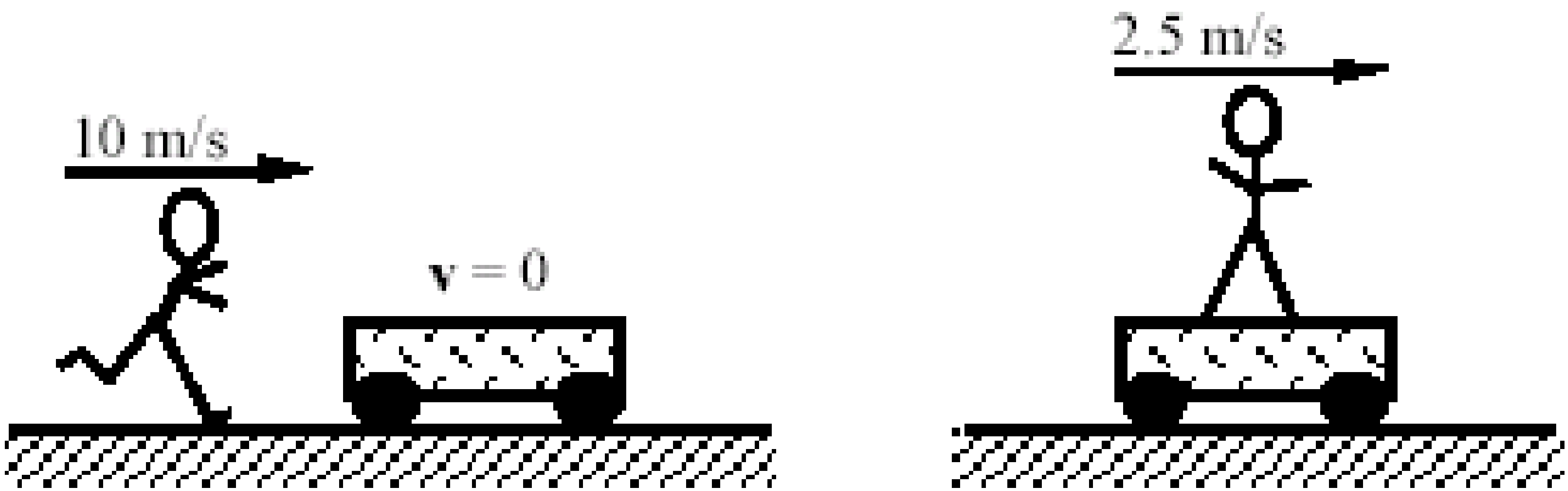
(a) What is the magnitude of velocity, $|\vec{v}_{2f}|$, of the second block?

(b) What is the angle, θ_2, after the collision that the trajectory of the second block makes as measured counter-clockwise from the x-axis?

A 50.0-kg boy runs at a speed of 10.0 m/s and jumps onto a cart as shown. The cart is initially at rest. If the speed of the cart with the boy on it is 2.50 m/s, what is the mass of the cart?

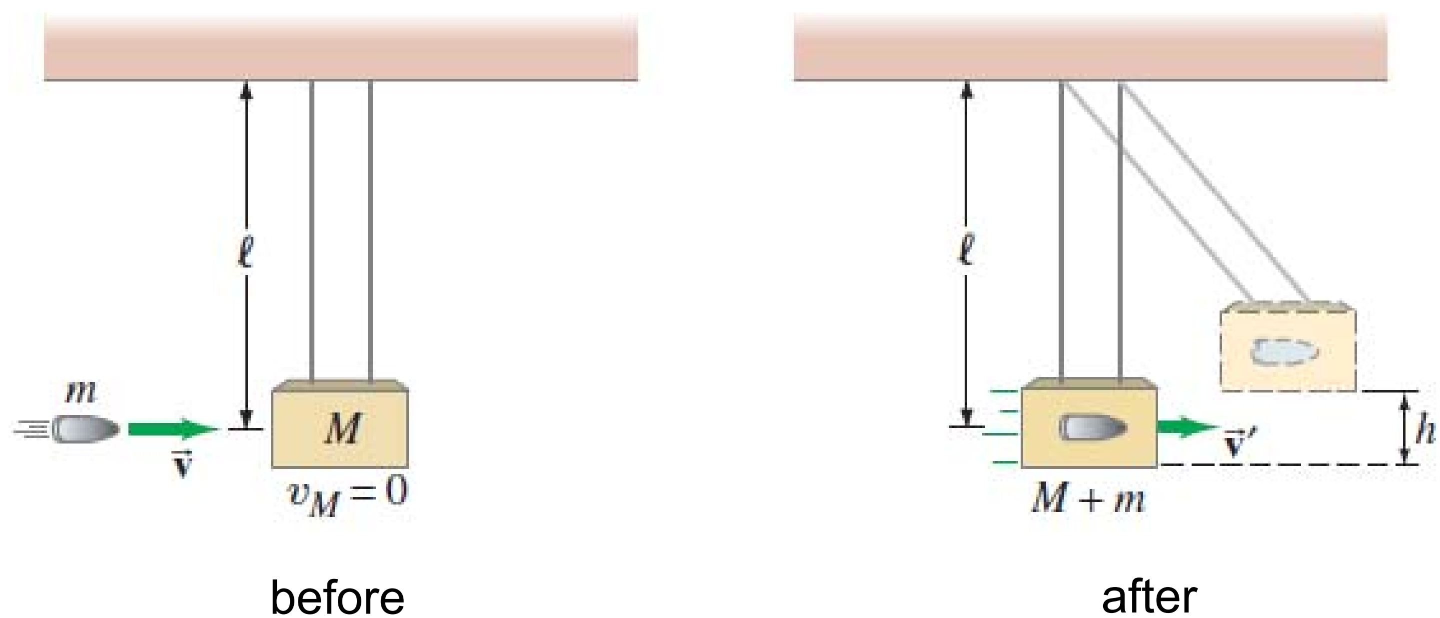
(1) 150 kg (2) 200 kg (3) 300 kg

(4) 175 kg (5) 100 kg



A bullet of mass m is fired with a velocity v_{bi}. The bullet collides and sticks inside a block of mass M initially at rest and hanging from the ceiling by a massless string of known length.

Find the maximum angle, θ, that the pendulum makes with respect to its equilibrium position.



Center of mass

The center-of-mass of many particles may be found by

$$\vec{r}_{\mathrm{cm}} = \frac{\displaystyle\sum_{i=1}^{N} m_i \vec{r}_i}{\displaystyle\sum_{i=1}^{N} m_i}$$

For a continuous distribution, the center-of-mass may be found via

$$\vec{r}_{\mathrm{cm}} = \frac{\displaystyle\int_0^M \vec{r}\, dm}{\displaystyle\int_0^M dm}$$

Substituting the density $\rho = \dfrac{dm}{dV}$ and integrating the lower term, we get

$$\vec{r}_{\mathrm{cm}} = \frac{1}{M} \iiint_{\mathcal{V}} \vec{r} \rho \, dV$$

Center of mass of moving particles

The position of the center-of-mass can change over time. The velocity of the center-of-mass is related to the momentum.

$$\frac{d}{dt}\vec{r}_{\mathrm{cm}} = \frac{\sum_{i=1}^{N} m_i \vec{v}_i}{\sum_{i=1}^{N} m_i} = \frac{\vec{p}}{M}$$

Rocket propulsion

For objects that do not have a constant mass (example, rockets with fuel depleting the initial mass), we consider an open system. Thus we cannot directly use Newton's second law, $\sum \vec{F} = m\vec{a}$. We can use the chain rule

$$\begin{aligned} \sum \vec{F_{\text{Ext}}} &= \frac{d\vec{p}}{dt} \\ &= \frac{d\left(m\vec{v}\right)}{dt} \\ &= m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} \end{aligned}$$

Remember that the total internal forces sum to zero if no external forces are acting on the rocket. Therefore,

$$0 = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$

and we find

$$m\frac{d\vec{v}}{dt} = -\vec{v}\frac{dm}{dt}$$

Rocket propulsion (cont.)

Also, the exhaust momentum is equal to the rockets momentum. Thus, the thrust of the rocket is given by

$$\vec{F} = -\vec{v}_{\text{exhaust}} \frac{dm}{dt}$$

where we have used Newton's third law.

The acceleration is then given by

$$\vec{a} = -\frac{\vec{v}_{\text{exhaust}}}{m} \frac{dm}{dt}$$

Using the definition $\vec{a} = \frac{d\vec{v}}{dt}$, we have

$$d\vec{v} = -\frac{\vec{v}_{\text{exhaust}}}{m} dm$$

Thus, $\int_{v_0}^{v} d\vec{v}' = -\vec{v}_{\text{exhaust}} \int_{m_0}^{m} \frac{dm'}{m'}$ ⇨ $\boxed{\vec{v} - \vec{v}_0 = -\vec{v}_{\text{exhaust}} \ln\left(\frac{m}{m_0}\right)}$