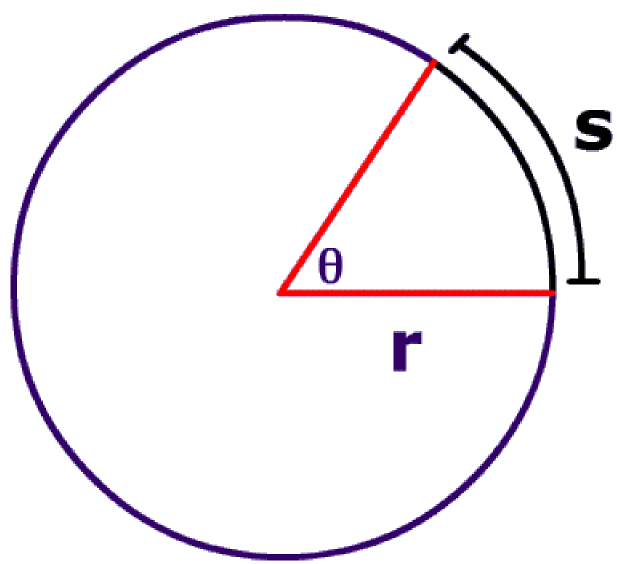
Angular position and arc length

The value of the polar angle, θ, (in radians) is equal to the arc length, s, divided by the radius, r,

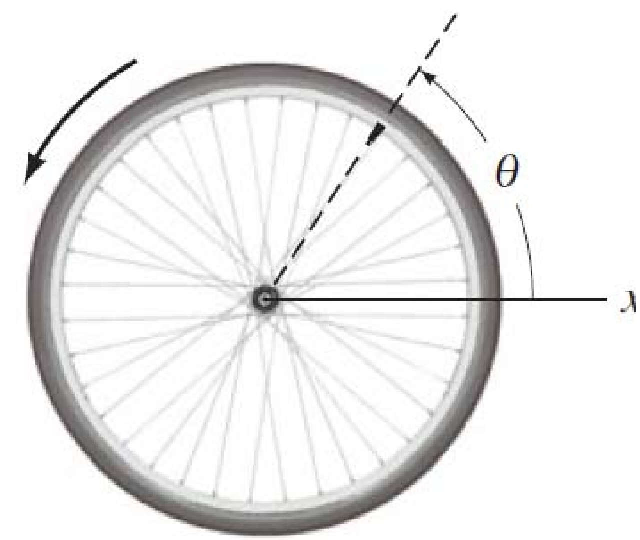
$$\theta = \frac{s}{r}$$



Angular velocity

Just like we learned with velocity, the angular velocity is given by

$$\omega = \frac{d\theta}{dt}$$



The dimensions are radians per time, where the dimensions of linear velocity are length per time.

For a rotating disk, the instantaneous linear velocity of a part of the disk far from the axis of rotation is greater than a part of the disk near the axis of rotation. The angular velocity of the two parts are equal (they sweep the same angle over the same amount of time).

Angular velocity (cont.)

The magnitude of the linear velocity is related to the magnitude of the angular velocity,

$$v = r\omega$$

The angular velocity is a vector and follows the right-hand-rule (RHR). This is because the rotation occurs in a plane, where planes are given an orientation in space by the normal vector.

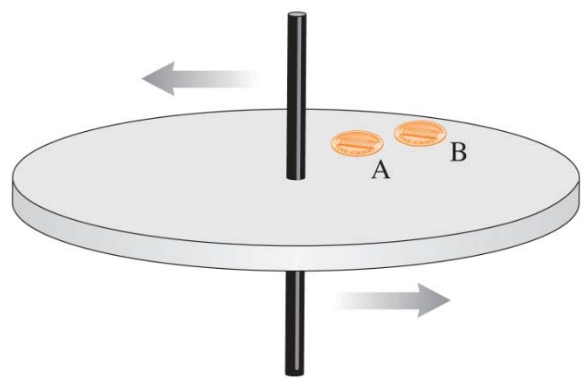
The rotation can be in two possible directions, which is defined by the corresponding normal vector of the plane. Curl your fingers along the rotating parts and your thumb points in the direction of $\vec{\omega}$.

Two coins rotate on a turntable. Coin B is twice as far from the axis as coin A.

A. The angular velocity of A is twice that of B.

B. The angular velocity of A equals that of B.

C. The angular velocity of A is half that of B.

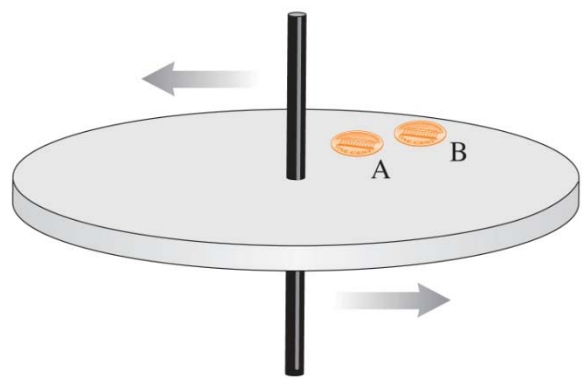


Two coins rotate on a turntable. Coin B is twice as far from the axis as coin A.

A. The speed of A is twice that of B.

B. The speed of A equals that of B.

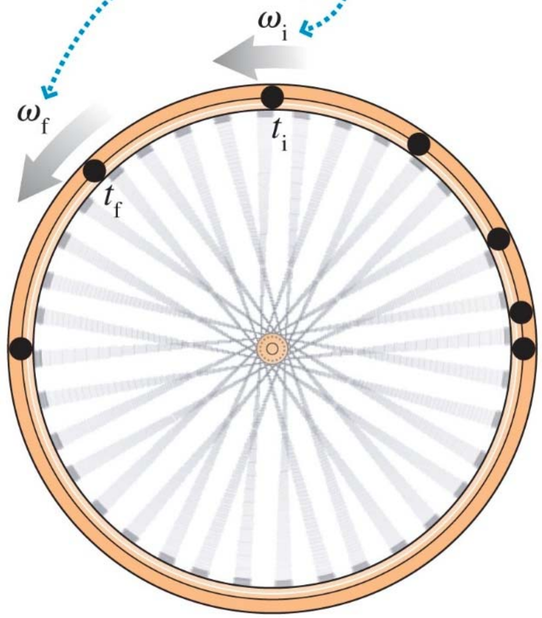
C. The speed of A is half that of B.



Angular acceleration

Similar to the linear acceleration, the angular acceleration is given by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$



The dimensions of angular acceleration are radians per time squared, where the dimensions of linear acceleration are length per time squared.

The magnitude of the linear acceleration is related to the angular acceleration,

$$a = r\alpha$$

Constant angular acceleration

(similarities to constant linear acceleration)

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

A wheel with a 0.10-m radius is rotating at 30 rev/s. It then slows uniformly to 15 rev/s over a 3.0 second interval. What is the angular acceleration of a point on the wheel?

Motion under a time-dependent angular acceleration

Assume we know the time dependence of the angular acceleration, $\alpha(t)$

The angular velocity can be calculated from the acceleration knowing $\alpha = \frac{d\omega}{dt}$ which can be rewritten to give

$$\omega(t) = \omega_0 + \int_0^t \alpha(t')\, dt'$$

The position can be calculated from the velocity knowing $\omega = \frac{d\theta}{dt}$, which can be rewritten to give

$$\theta(t) = \theta_0 + \int_0^t \omega(t')\, dt'$$

Finding the velocity and acceleration from a time dependent angular postion

Assume instead that we know the time dependence of the angular position, $\theta(t)$

The time-dependent angular velocity can be calculated by finding the slope of $\theta(t)$

$$\omega = \frac{d\theta}{dt}$$

The time-dependent angular acceleration can be calculated by finding the slope of $\omega(t)$

$$\alpha = \frac{d\omega}{dt}$$

Revisiting radial acceleration

Remember that we initially learned the radial acceleration formula for an object moving in a circle as

$$a_{\text{rad}} = \frac{v^2}{r}$$

Using $v = \omega r$ we can rewrite the radial acceleration as

$$a_{\text{rad}} = r\omega^2$$

Moment of inertia

The moment of inertia for a collection of N point masses rotating about a principal axis is given by

$$I = \sum_{i=1}^{N} r_i^2 m_i$$

For a continuous approximation, the moment of inertia can be written in integral form,

$$I = \int_0^M r^2 \, dm$$

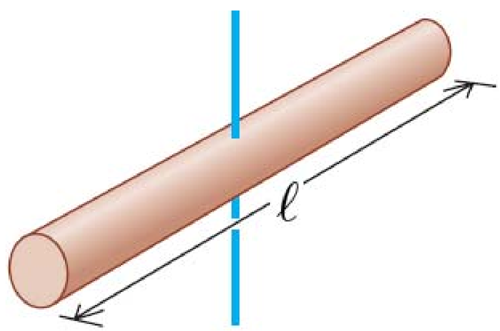
For an object of known density, the moment of inertia equation can be rewritten as

$$I = \iiint_{\mathcal{V}} r^2 \rho \, dV$$

where

$$\rho = \frac{dm}{dV}$$

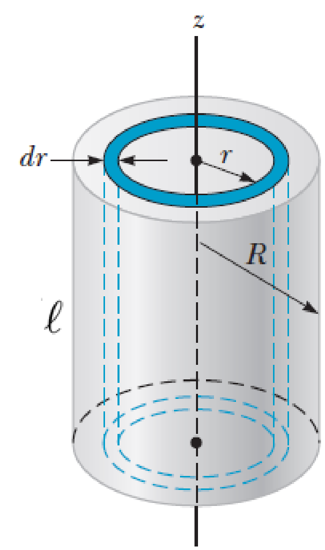
Example: Moment of inertia for a thin rod with a uniform mass density of λ



$$
\begin{aligned}
I_{\mathrm{CM}} &= \int_0^m x^2\, dm' \\
&= \int_{-\ell/2}^{\ell/2} x^2 \lambda\, dx \\
&= \frac{1}{3}\lambda x^3 \Big|_{-\ell/2}^{\ell/2} \\
&= \frac{\lambda}{3}\left[\left(\frac{\ell}{2}\right)^3 - \left(\frac{-\ell}{2}\right)^3\right] \\
&= \lambda\frac{\ell^3}{12} \\
&= m\frac{\ell^2}{12}
\end{aligned}
$$

Example: Uniform cylinder or disk

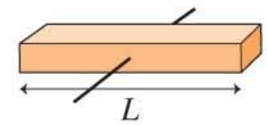
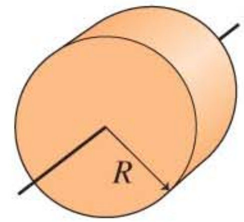
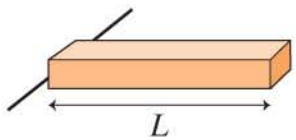
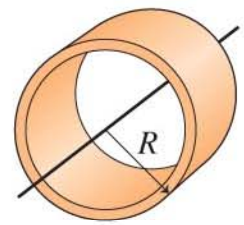
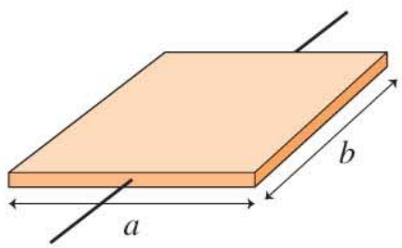
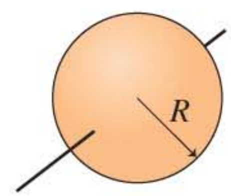
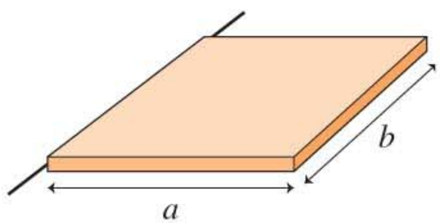
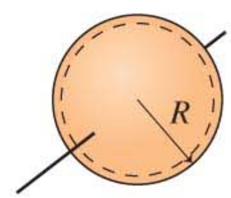
$$
\begin{aligned}
I_{\mathrm{CM}} &= \int_0^m r^2\, dm' \\
&= \int_0^{\ell} \int_0^{2\pi} \int_0^{R} \left(r^2 \rho\right)\, r\, dr\, d\theta\, dz \\
&= z\Big|_0^{\ell}\, \theta\Big|_0^{2\pi}\, \frac{\rho}{4} r^4\Big|_0^{R} \\
&= \frac{\rho}{4} (\ell - 0)\, (2\pi - 0)\, \left(R^4 - 0\right) \\
&= \frac{1}{2} \pi \rho \ell R^4
\end{aligned}
$$



We know the density of a cylinder to be $\rho = \dfrac{m}{\pi \ell R^2}$, and therefore,

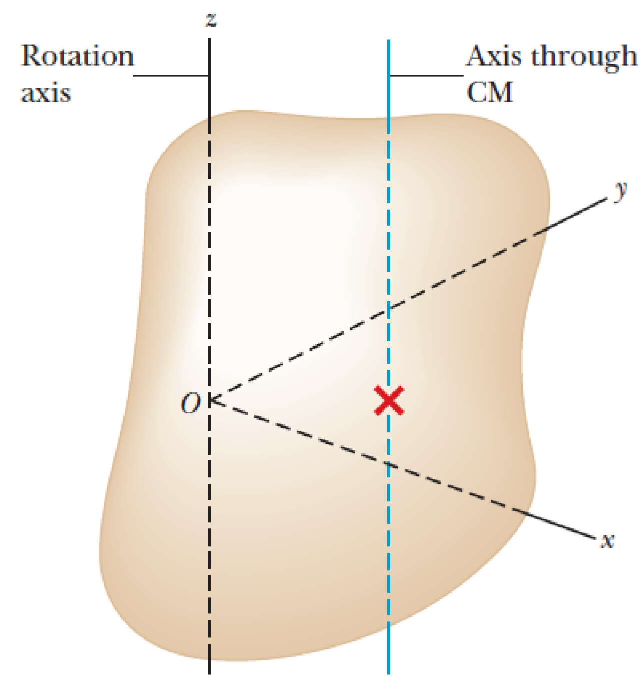
$$I_{\mathrm{CM}} = \frac{1}{2} m R^2$$

Moments of Inertia of Common Shapes

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod (of any cross section), about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod (of any cross section), about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

Parallel axis theorem

To calculate the moment of inertia, we must calculate the moment of inertia about the center-of-mass of the object, I_{CM}. Then, we use the parallel axis theorem to calculate the total moment of inertia. The parallel axis theorem is given by



$$I = I_{CM} + Md^2$$

<u>Derivation</u>

A rigid body in three dimensions can be defined using Cartesian coordinates (x,y,z), with point masses at (x_i,y_i,z_i). The moment of inertia about the center-of-mass for a body rotating about the z-axis is given by,

$$I = \sum_{i}^{N} m_i \left(x_i^2 + y_i^2\right)$$

Parallel axis theorem (cont.)

If we move our axis of rotation away from the center-of-mass (still taken as the origin), still perpendicular to the *z*-axis, through another point in the *xy*-plane, then the moment of inertia becomes,

$$I = \sum_{i}^{N} m_i \left[(x_i - x_0)^2 + (y_i - y_0)^2 \right]$$

$$= \sum_{i}^{N} m_i \left(x_i^2 + y_i^2 \right) - 2x_0 \sum_{i}^{N} m_i x_i - 2y_0 \sum_{i}^{N} m_i y_i + \left(x_0^2 + y_0^2 \right) \sum_{i}^{N} m_i$$

$$= \sum_{i}^{N} m_i \left(x_i^2 + y_i^2 \right) + Md^2$$

where $\sum_{i}^{N} m_i x_i = \sum_{i}^{N} m_i y_i = 0$ because we were at the center of mass.

Also, $\sum m_i = M$ and $d^2 = x_0^2 + y_0^2$.

Moment of inertia is

A. the rotational equivalent of mass.

B. the point at which all forces appear to act.

C. the time at which inertia occurs.

D. an alternative term for moment arm.

You have two uniform solid cylinders of the same material. Cylinder 2 has twice the radius of cylinder 1 but the same length. By what *factor* does the CM moment of inertia (about symmetry axis) I_2 of cylinder 2 exceed the moment of inertia I_1 of cylinder 1.

(a) 4

(b) 8

(c) 16

(d) their moments of inertia are the same.

Rotational kinetic energy

If a round object rolls across the floor, it has both linear motion and rotational motion. Thus, there is kinetic energy associated with both the linear and rotational motion.

The velocity, v_i, of a mass element of a rotating rigid object of mass m_i is moving with a relative velocity, v_i', with respect to the linear velocity of the rigid object's center-of-mass, v_{CM}. The kinetic energy of the mass element can then be written as

$$
\begin{aligned}
K_i &= \frac{1}{2} m_i v_i^2 \\
&= \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \\
&= \frac{1}{2} m_i \left(\vec{v}_i' + \vec{v}_{\mathrm{CM}}\right) \cdot \left(\vec{v}_i' + \vec{v}_{\mathrm{CM}}\right) \\
&= \frac{1}{2} m_i \left(\vec{v}_i' \cdot \vec{v}_i' + \vec{v}_{\mathrm{CM}} \cdot \vec{v}_{\mathrm{CM}} + 2\vec{v}_i' \cdot \vec{v}_{\mathrm{CM}}\right) \\
&= \frac{1}{2} m_i \left(v_i'^2 + v_{\mathrm{CM}}^2 + 2\vec{v}_i' \cdot \vec{v}_{\mathrm{CM}}\right)
\end{aligned}
$$

Rotational kinetic energy (cont.)

We can sum the mass elements to find the total kinetic energy of the object,

$$
\begin{aligned}
K &= \sum_i^N K_i \\
&= \sum_i^N \frac{1}{2} m_i \left(v_i'^2 + v_{\mathrm{CM}}^2 + 2\vec{v}_i' \cdot \vec{v}_{\mathrm{CM}} \right) \\
&= \frac{1}{2} \left(\sum_i^N m_i v_i'^2 \right) + \frac{1}{2} \left(\sum_i^N m_i \right) v_{\mathrm{CM}}^2 + \left(\sum_i^N m_i \vec{v}_i' \right) \cdot \vec{v}_{\mathrm{CM}} \\
&= \frac{1}{2} \left(\sum_i^N m_i v_i'^2 \right) + \frac{1}{2} \left(\sum_i^N m_i \right) v_{\mathrm{CM}}^2
\end{aligned}
$$

where the last term was removed because adding the velocities of all mass elements about the center-of-mass in the reference frame relative to the center-of-mass is equal to zero.

Rotational kinetic energy (cont.)

Previously, we discovered that the moment of inertia of a rotating rigid object is equal to the sum of the mass elements multiplied by their relative positions squared,

$$I = \sum m_i r_i^2$$

Substituting $v_i' = \omega r_i'$ into our kinetic energy equation, we find

$$K = \frac{1}{2}\left(\sum_i^N m_i r_i^2 \omega^2\right) + \frac{1}{2}\left(\sum_i^N m_i\right) v_{\mathrm{CM}}^2$$

$$= \underbrace{\boxed{\frac{1}{2} I\omega^2}}_{K_{\mathrm{rot}}} + \underbrace{\boxed{\frac{1}{2} M v_{\mathrm{CM}}^2}}_{K_{\mathrm{trans}}}$$

where M is the total mass of the rigid object. If the rolling object is an object with nice symmetry with a constant edge at radius R such as a sphere, cylinder, hoop, etc., then

$$K = \frac{1}{2} I \frac{v_{\mathrm{CM}}^2}{R^2} + \frac{1}{2} M v_{\mathrm{CM}}^2$$

A wheel with a weight of 400 N comes off a moving truck and rolls without slipping along a highway. At the bottom of a hill it is rotating at an angular velocity of 25.0 rad/s . The radius of the wheel is $R = 0.50$ m and its moment of inertia about its rotation axis is $0.80\ MR^2$ (M is the mass of the wheel). Assume that there is no slipping.

(a) Determine the rotational kinetic energy of the wheel at the bottom of the hill immediately after it comes off the truck.

(b) Determine the initial speed of the wheel's center-of-mass at the bottom of the hill.

(c) Determine the translational kinetic energy at the bottom of the hill immediately after it comes off the truck.

(d) Determine the maximum height that the wheel reaches when it rolls up the hill.

(e) If the rolling object was actually a hoop with $I_{hoop} = MR^2$ instead of a wheel, would the height that the hoop reaches be greater than, less than, or the same as the height that the wheel reached?

Rotational kinetic energy

The kinetic energy of rotating point mass is given by

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(r\omega\right)^2 \\ &= \frac{1}{2}mr^2\omega^2 \end{aligned}$$

For several points rotating, the total kinetic energy is the sum of the kinetic energy of all point masses,

$$K = \sum_{i}^{N} \frac{1}{2} m_i r_i^2 \omega_i^2$$

Rotational kinetic energy (cont.)

We know that points in a rigid body rotate at the same angular velocity $\omega_i = \omega_j$, where $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$. So,

$$K = \sum_i^N \frac{1}{2} m_i r_i^2 \omega^2$$
$$= \frac{1}{2} \left(\sum_i^N m_i r_i^2 \right) \omega^2$$

where we define the moment of inertia of the object as

$$I = \sum_i^N m_i r_i^2$$

Thus, we can rewrite the rotational kinetic energy as

$$K = \frac{1}{2} I \omega^2$$

A wheel with a weight of mg starts at rest on top of a hill. The radius of the wheel is R and its moment of inertia about its rotation axis is $\frac{3}{4}mR^2$ (m is the mass of the wheel). After the wheel rolls from the top of the hill to the bottom a height h below. Assume that there is no slipping.

(a) Determine the rotational kinetic energy of the wheel at the bottom of the hill.

(b) Determine the translational kinetic energy at the bottom of the hill.

(c) Determine the final angular velocity.