Chapter 13: universal gravitation

- Newton’s Law of Gravitation
- Weight
- Gravitational Potential Energy
- The Motion of Satellites
- Kepler’s Laws and the Motion of Planets
- Spherical Mass Distributions
- Apparent Weight and the Earth’s Rotation
- Black Holes

Newton's law of universal gravitation

Newton's law of universal gravitation: every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

$$F_g = G\frac{m_1 m_2}{r^2}$$

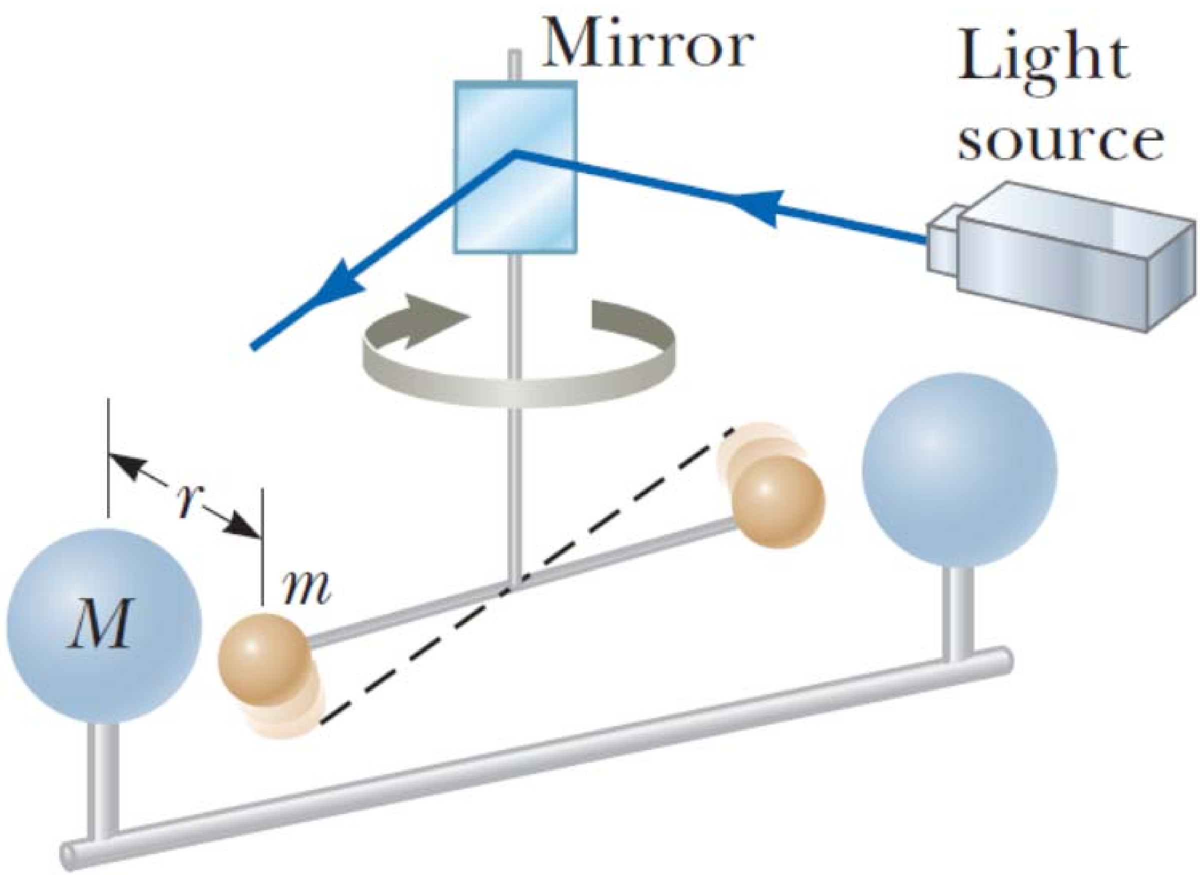
where $G \approx 6.674\times10^{-11}$ N m^2 kg^{-2}.

Note that the masses are assumed to be point masses with no significant size relative to the distance the two points. The force of gravity is always an attractive force, where the gravitational force vectors of two particles are always pointed towards each other.

For a spherically symmetric mass distribution, the force on a particle outside the distribution is the same as that for a point charge.

Cavendish experiment

Henry Cavendish (1731–1810) measured the universal gravitational constant in an important 1798 experiment.

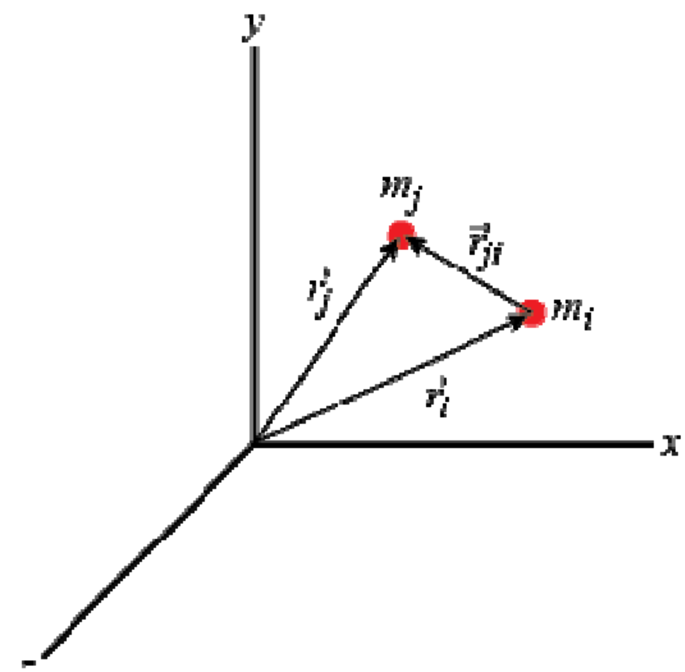


Weight

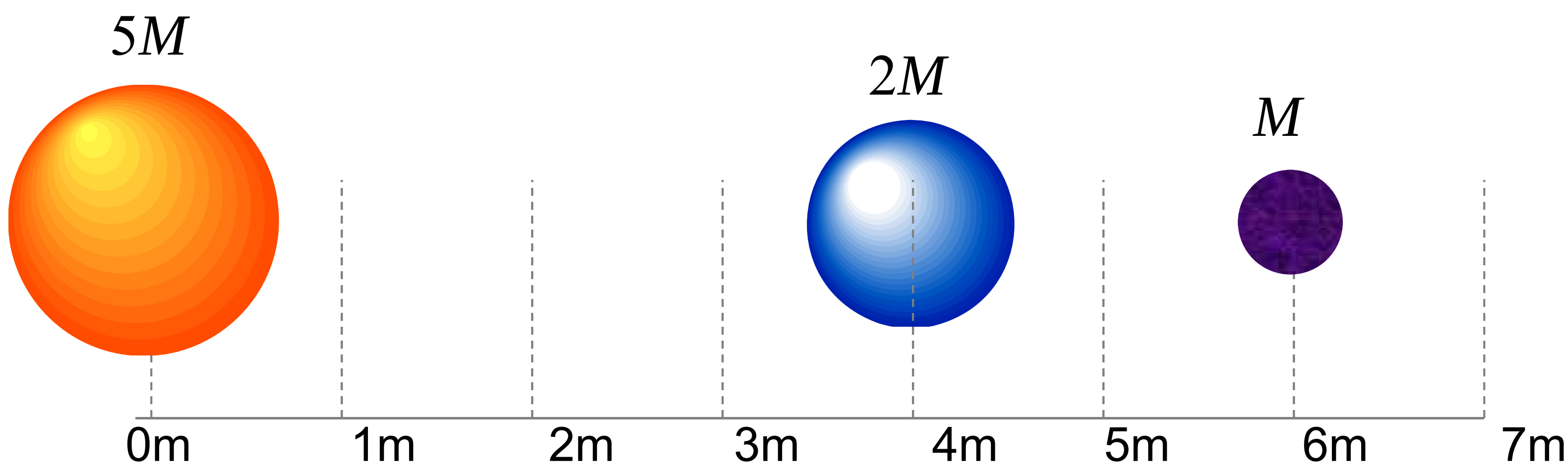
The weight of a body with mass m_i is the total gravitational force exerted on the body by all other bodies in the universe. If we sum up the weight of a single particle from the gravitational interaction of all other particles,

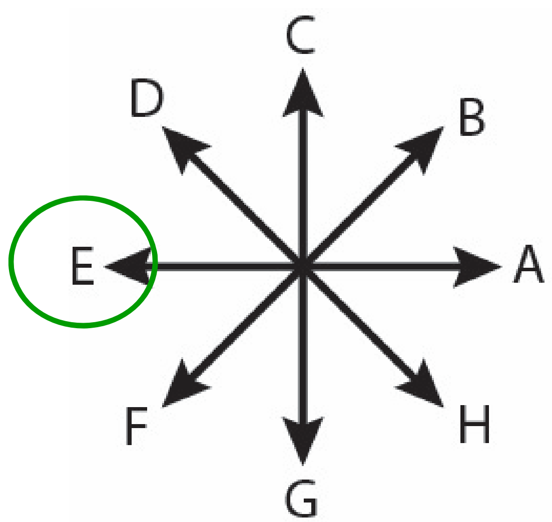
$$\vec{w}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N-1} \vec{F}_{j\,\mathrm{on}\,i}$$

$$= -m_i G \sum_{\substack{j=1 \\ j\neq i}}^{N-1} \frac{m_j}{r_{ji}^2}\hat{r}_{ji} \quad \text{where} \quad \vec{r}_{ji} = \vec{r}_j - \vec{r}_i$$

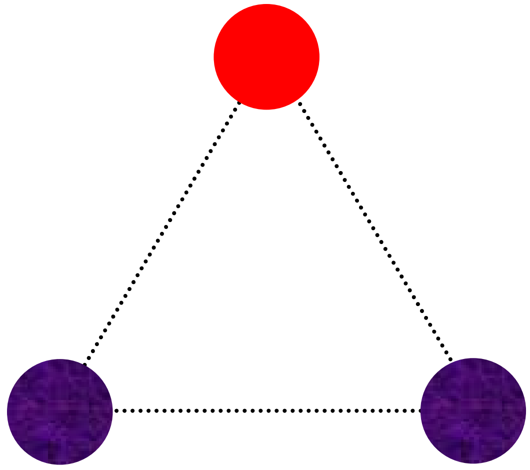


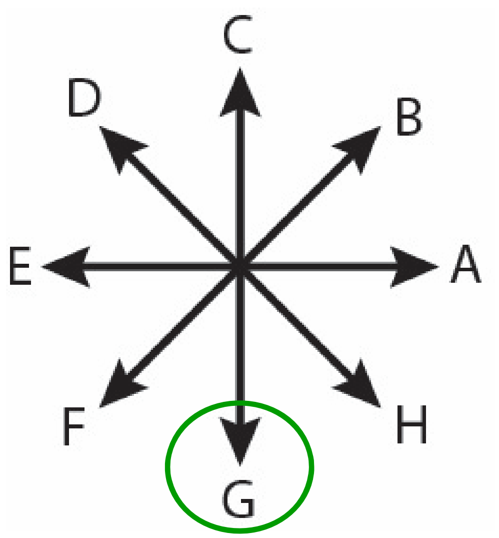
Three balls of mass *M*, 2*M*, and 5*M* are arranged as shown below. What is the direction of the net gravitational force ***on the 2M mass*** due to *M* and 5*M*?





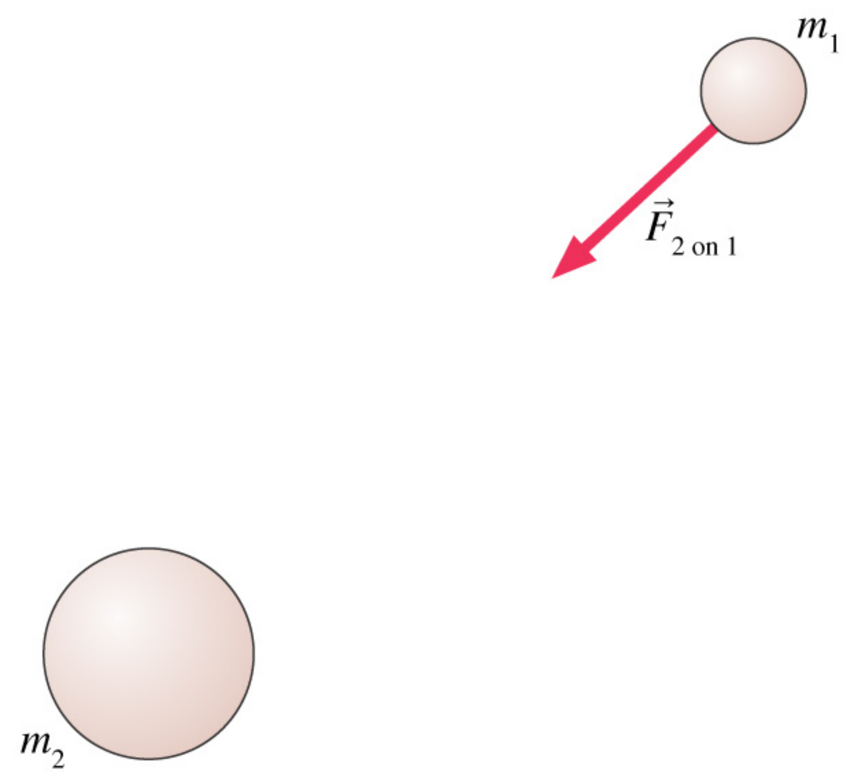
Three balls of ***<u>equal mass M</u>*** are arranged at the corners of a triangle as shown below. All sides of the triangle are equal in length. What is the direction of the net gravitational force on the top (red) mass due to the bottom two masses?



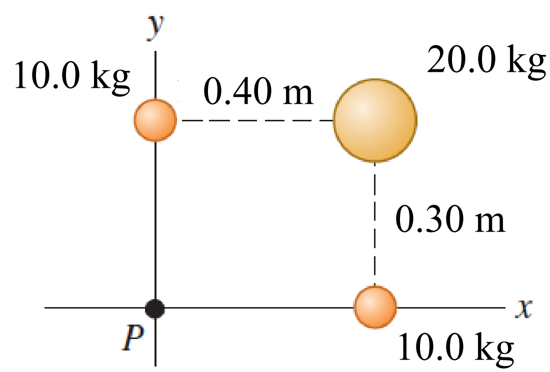


The figure shows a binary star system. The mass of star 2 is twice the mass of star 1. Compared to $\vec{F}_{1 \text{ on } 2}$, the magnitude of the force $\vec{F}_{2 \text{ on } 1}$ is

1. one quarter as much.
2. half as much.
3. twice as much.
4. four times as much.
5. the same amount.



Three uniform spheres are in deep outer space and fixed at the positions shown in the diagram. A 0.100 kg particle is placed at point *P*.



(a) Determine the magnitude of the force on the 0.100 kg particle at point *P*.

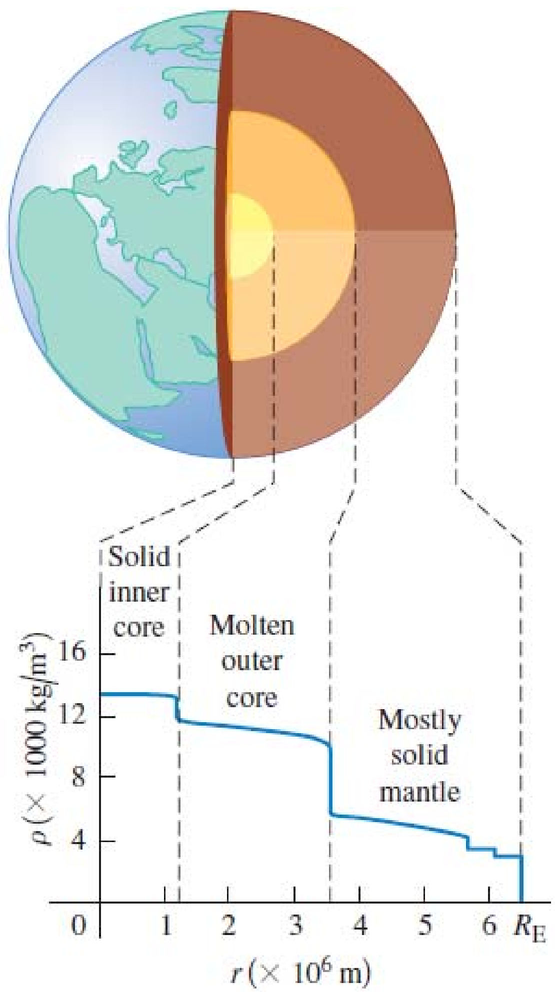
(b) Determine the direction of the force on the 0.100 kg particle at point *P*.

Weight on Earth

Most of the particles in the Universe are really far away, which make their contribution to the weight negligible. On the surface of Earth, even other objects within our solar system are so far away that we can consider their force contributions to be negligible.

The Earth may also be approximated as a spherically symmetric object, in which case the weight is simply

$$\vec{w}_{\text{object}} \approx -G\frac{m_{\text{object}}m_E}{R_E^2}\hat{r}$$



Gravitational accel. near the Earth's surface

The gravitational acceleration at the surface of the Earth is

$$\begin{aligned}\vec{g} &= \frac{\vec{F}_g}{m} \\ &= -G\frac{m_E}{R_E^2}\hat{r} \\ &\approx -\left(9.8\,\mathrm{m/s^2}\right)\hat{r}\end{aligned}$$

where $m_E = 5.97 \times 10^{24}\mathrm{kg}$ and $R_E = 6370\,\mathrm{km}$.

The gravitational acceleration at some height *h* is then

$$\vec{g}\,(h) = -G\frac{m_E}{(R_E + h)^2}\hat{r}$$

Gravitational accel. near the Earth's surface (cont.)

For $h < R_E$ we can do a binomial expansion of the denominator about h/R_E, which gives

$$\vec{g}(h) = -G\frac{m_E}{R_E^2}\hat{r}\left(1 - 2\frac{h}{R_E} + 3\frac{h^2}{R_E^2} - 4\frac{h^3}{R_E^3} + \cdots\right)$$

When $h << R_E$, $\boxed{\vec{g} \approx -G\frac{m_E}{R_E^2}\hat{r}}$

*To calculation the above expression, we used the binomial expansion for negative integer powers, which is

$$(1+x)^{-q} = 1 - qx + \frac{1}{2!}q\,(q+1)\,x^2 - \frac{1}{3!}q\,(q+1)\,(q+2)\,x^3 + \cdots$$

Work due to universal gravitation

From earlier we learned the relationship between the work due to a force and the distance

$$W = \int \vec{F} \cdot d\vec{s}$$

If we are concerned with being outside a spherical mass, and we are interested in great distances such that the force is not approximated as a constant, and displacements along the radial direction,

$$\begin{aligned} W_g &= -Gm_1m_2 \int_{r_A}^{r_B} \frac{dr}{r^2} \\ &= Gm_1m_2 \left(\frac{1}{r_B} - \frac{1}{r_A} \right) \end{aligned}$$

Gravitational potential energy

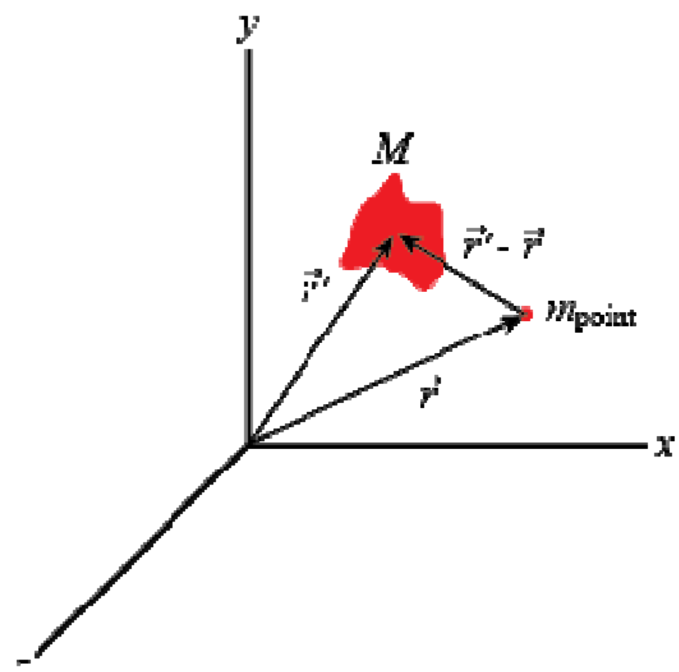
We know that the work due to gravity is related to the potential energy, $W_g = -\Delta U_g$. Setting $r_A \to \infty$, we find

$$U_g = -G\frac{m_1 m_2}{r}$$

In general for a small point mass and an object with any mass distribution, the gravitational potential energy from Newton's universal law of gravity is given by

$$U(r) = -Gm_{\text{point}} \int_0^M \frac{dm'}{|\vec{r}\,' - \vec{r}|}$$

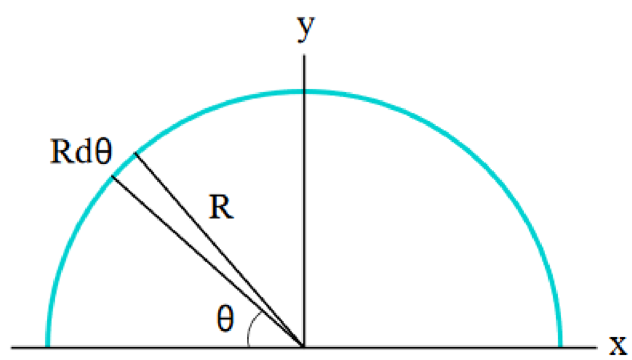
$$= -Gm_{\text{point}} \iiint_{\mathcal{V}} \frac{\rho(\vec{r}\,')}{|\vec{r}\,' - \vec{r}|} dV'$$



where m_{point} is the small point mass.

Example with gravitational potential energy

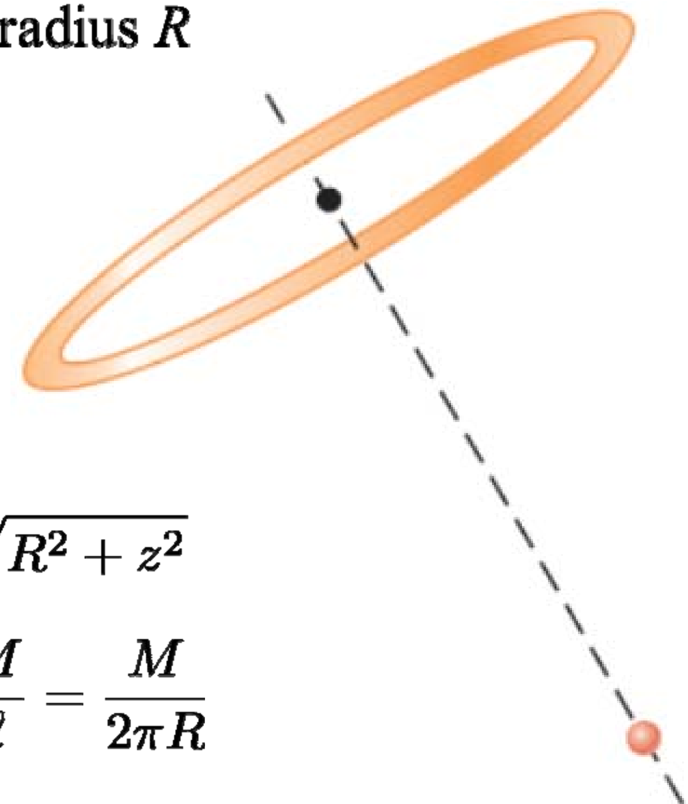
Determine the gravitational potential energy of a point mass m_p in a semicircle of uniform mass density with total mass M.



$$
\begin{aligned}
U_{\text{semi}} &= -Gm_p \int_0^M \frac{dm'}{r} \\
&= -Gm_p \int_\ell \frac{\lambda}{r} ds \\
&= -Gm_p\lambda \int_0^\pi \frac{R\,d\theta}{R} \\
&= -Gm_p\lambda\pi \quad \text{where} \quad \lambda = \frac{M}{\ell} = \frac{M}{\pi R} \\
&= -\frac{Gm_pM}{R}
\end{aligned}
$$

2nd example with gravitational potential energy

Determine the gravitational potential energy of a point mass m_p that is a distance z (along the principal axis) away from the center of a ring with a uniform mass density, total mass M, and radius R



$$
\begin{aligned}
U_{\text{ring}} &= -Gm_p \int_0^M \frac{dm'}{r} \\
&= -Gm_p \int_\ell \frac{\lambda}{r} ds \\
&= -Gm_p\lambda \int_0^{2\pi} \frac{R\,d\theta}{r} \quad \text{where } r = \sqrt{R^2+z^2} \\
&= -Gm_p\lambda \frac{2\pi R}{\sqrt{R^2+z^2}} \quad \text{where } \lambda = \frac{M}{\ell} = \frac{M}{2\pi R} \\
&= -\frac{Gm_pM}{\sqrt{R^2+z^2}}
\end{aligned}
$$

Satellites in orbit

- Remember that centripetal acceleration is $a = \frac{v_\perp^2}{r}$

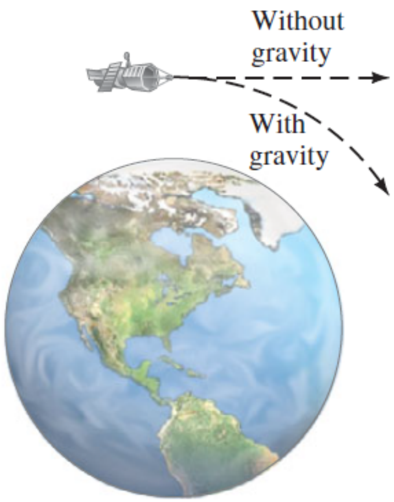
- The magnitude of the force is $F = ma_{\text{rad}}$ which gives $ma_{\text{rad}} = m\frac{v_\perp^2}{r}$

- Newtons law of gravity is $F_g = G\frac{mm_E}{r^2}$

- Determine the sum of the forces, $\sum F = G\frac{mm_E}{r^2} = ma$

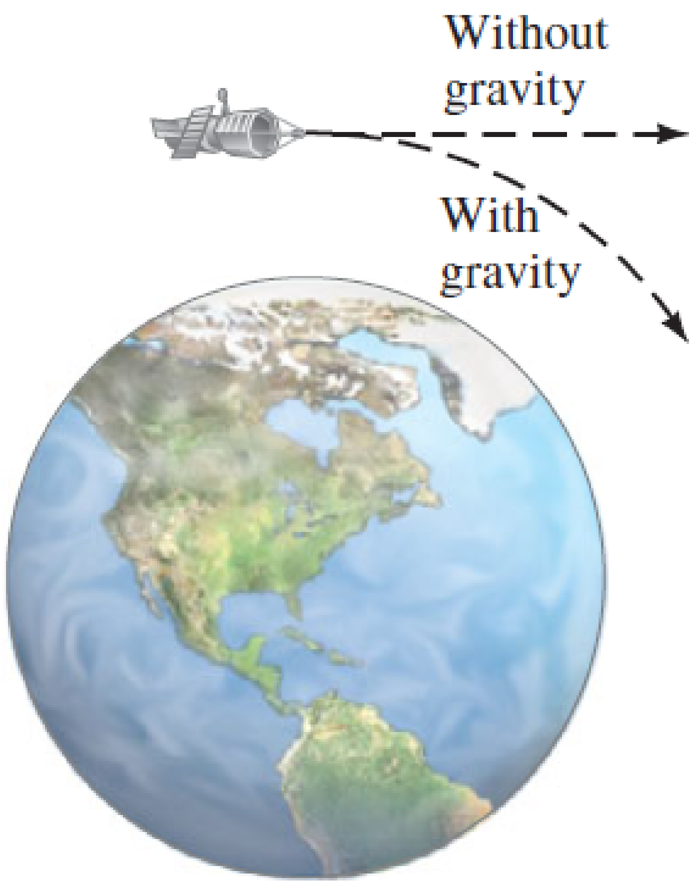
- The only acceleration is a_{rad}, so $G\frac{mm_E}{r^2} = m\frac{v_\perp^2}{r}$ (A)

- We know the circumference of a circle and that $v_\perp$ is a constant velocity, therefore $v_\perp = \frac{\Delta s}{\Delta t} = \frac{2\pi r}{T}$ (B)



- Find the radius of orbit by plugging in (B) into (A) and solving for r gives the radius of the orbit, $r_{\text{orbit}} = \sqrt[3]{G\frac{m_E T^2}{4\pi^2}}$.

Work and circular orbits



How much work is done on a satellite to have it orbit 1 period?

NO WORK DONE!!!

$v_\perp = \text{constant}$ and therefore $F_\perp = 0$

Thus, $W = \vec{F} \cdot \vec{d}$

$= (ma)\ (2\pi r) \cos 90°$

$= 0$

Also note the total energy of circular orbit is

$$E = K + U_g = \frac{1}{2}mv^2 - G\frac{mM}{r} = \frac{1}{2}m\left(G\frac{M}{r}\right) - G\frac{mM}{r} = -\frac{1}{2}G\frac{mM}{r}$$

You wish to put a 1000-kg satellite into a circular orbit 300 km above the earth's surface.

(a) What speed will it have?
(b) What period will it have?
(c) How much total work must be done to the satellite to put it in orbit starting from rest on the surface of Earth?
(d) How much additional work would have to be done to make the satellite escape the earth?

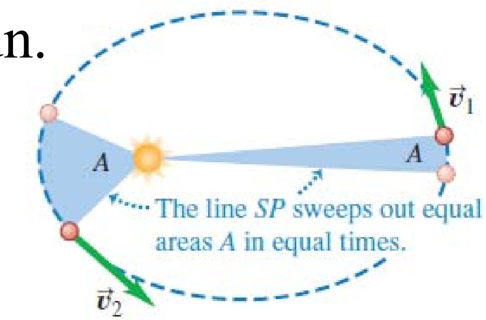
[Universal gravitational constant, $G = 6.67\times10^{-11}$ N·m²/kg²; mass of the Earth, $M_E = 5.98\times10^{24}$ kg; radius of the Earth, $R_E = 6380$ km]

Kepler's laws of planetary motion

Kepler's first law: planets are orbiting the sun in a path described as an ellipse, where the sun is at one focus of the ellipse.

Kepler's second law: The radius vector sweeps equal areas in equal times for a planet in motion around the sun.

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$



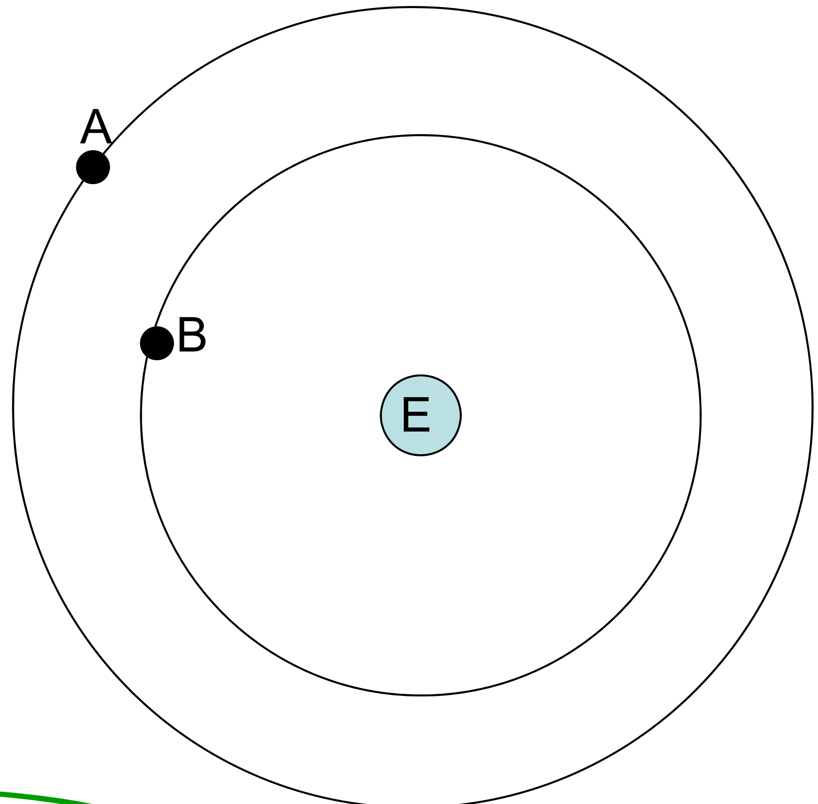
Kepler's third law: The period of a planet's orbit squared is proportional to its average distance from the sun cubed.

$$T^2 \propto r_{\mathbf{avg}}^3$$

The circular orbit solution for the period can be extended to an elliptical orbit by replacing the "*r*" by the semi-major axis "*a*"

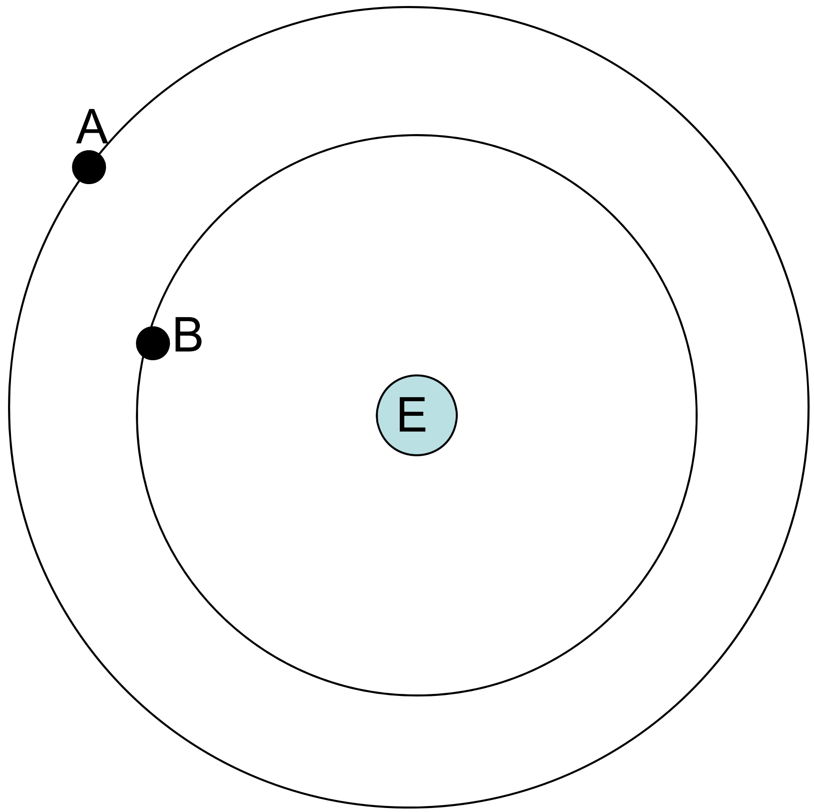
$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}}$$

Which satellite orbits with a longer period *T* around earth?



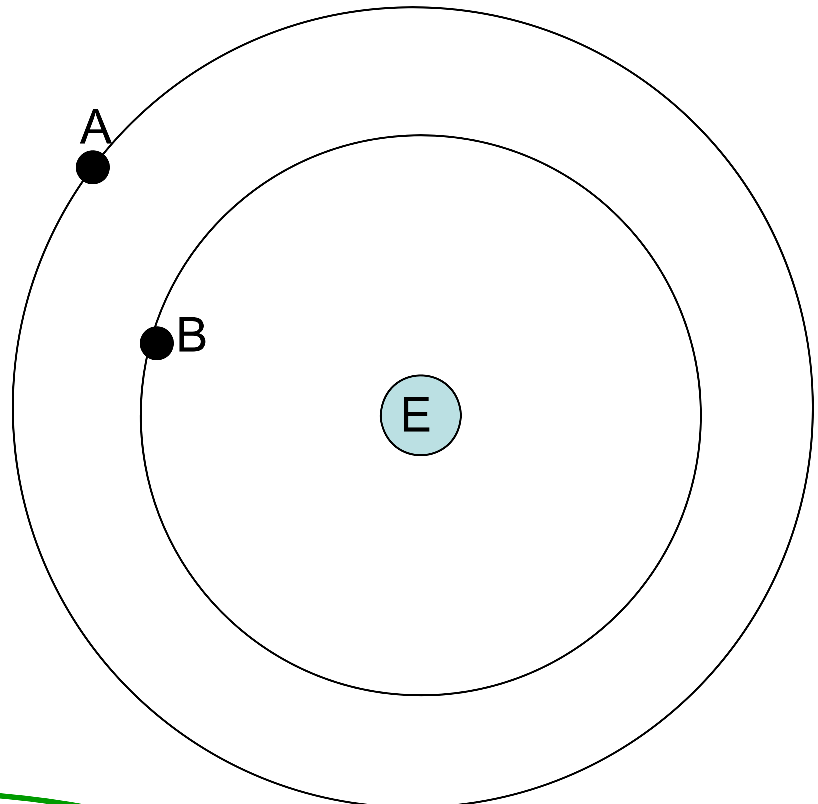
1. Satellite A
2. Satellite B
3. Not enough information

Both satellites have the same mass. Which satellite has the larger kinetic energy *K*?



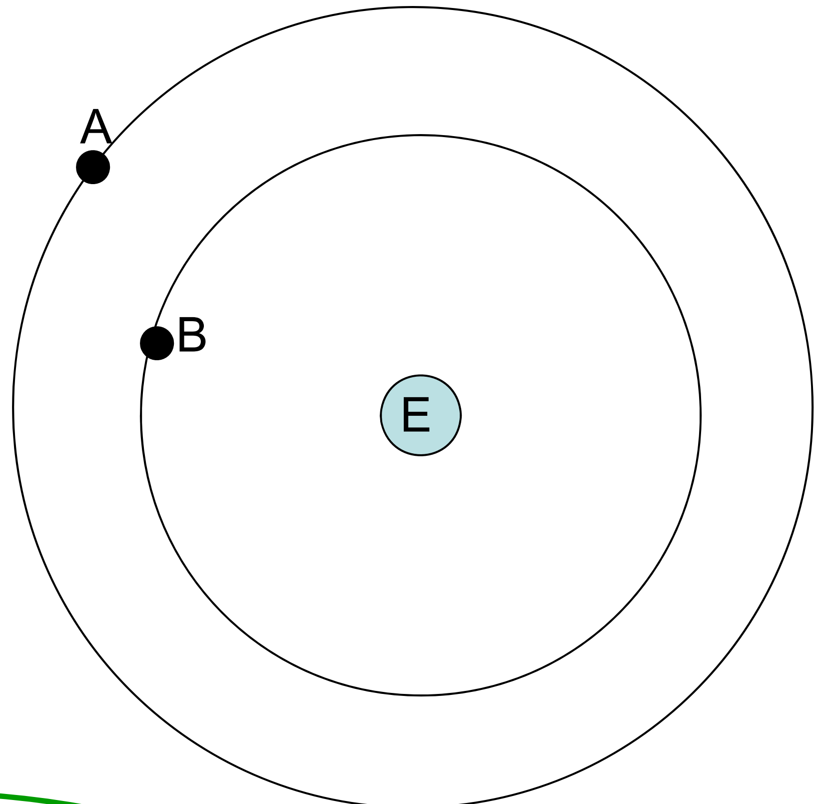
1. Satellite A
2. Satellite B
3. Not enough information

Both satellites have the same mass. Which satellite has the greatest gravitational potential energy U_g?



1. Satellite A
2. Satellite B
3. Not enough information

Both satellites have the same mass. Which satellite has the greatest total orbital energy E?



1. Satellite A
2. Satellite B
3. Not enough information

Apparent weight on a rotating planet

From Newton's second law, the sum of all forces is equal to the mass multiplied by the acceleration. An object on the surface of a spinning object can be in an accelerating reference frame, where

$$N - mg = -m\frac{v^2}{R} \quad \Longrightarrow \quad N = m\left(g - \frac{v^2}{R}\right) = mg_{\text{effective}}$$

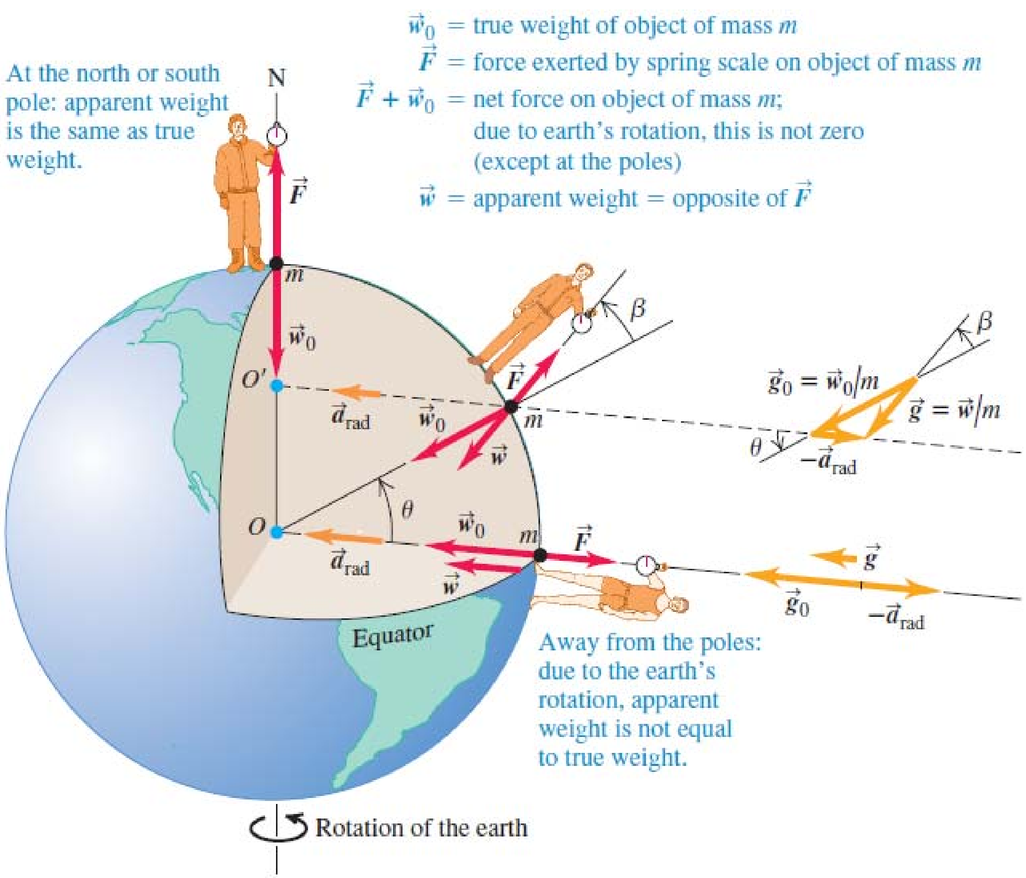


Table 13.1 Variations of g with Latitude and Elevation

Station	North Latitude	Elevation (m)	$g(\mathrm{m/s^2})$
Canal Zone	09°	0	9.78243
Jamaica	18°	0	9.78591
Bermuda	32°	0	9.79806
Denver, CO	40°	1638	9.79609
Pittsburgh, PA	40.5°	235	9.80118
Cambridge, MA	42°	0	9.80398
Greenland	70°	0	9.82534

Black holes

You can easily derive the escape velocity from a spherical object using the conservation of mechanical energy and taking the limits to be at the surface of object and at infinity,

$$v_{\text{escape}} = \sqrt{\frac{2GM}{R}} = R\sqrt{\frac{8}{3}\pi G\rho}$$

If the escape velocity (velocity required to escape the object) is greater than the speed of light, $v_{\text{escape}} > c$, then no known particle including light can escape the object.

The Schwarzschild radius is the radial displacement from the center of the object where

visible object

$$R_S = \frac{2GM}{c^2}$$

black hole

If a particle is beyond the horizon defined by this radius, then it can escape the gravitational pull. If the particle is inside this radius, then it cannot escape the gravitational pull of the object. Objects with a Schwarzschild radius are black holes!