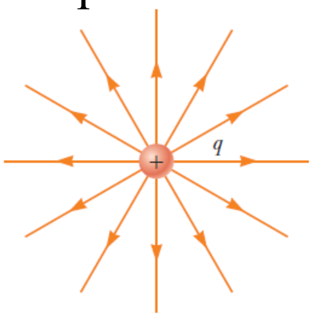
Chapter 22: Gauss's law

- Charge and electric flux
- Calculating electric flux
- Gauss's law
- Applications of Gauss's law
- Charges on conductors

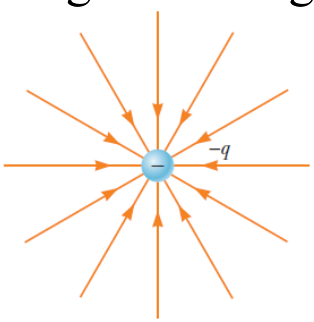
Useful concepts

Electric field lines

Electric field lines point away from positive charges.



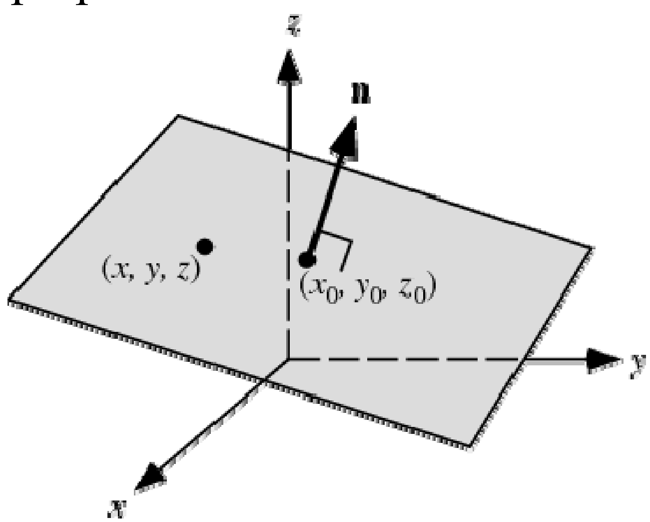
Electric field lines point toward negative charges.



The field strength decreases with increasing separation distance between lines.

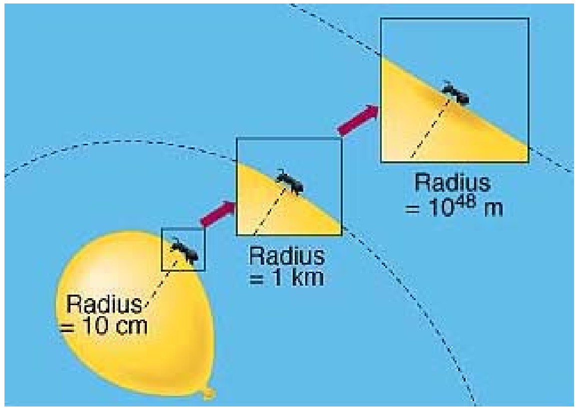
Normal vector

The orientation of a planar surface can be described by a vector pointing outward and perpendicular to the surface.



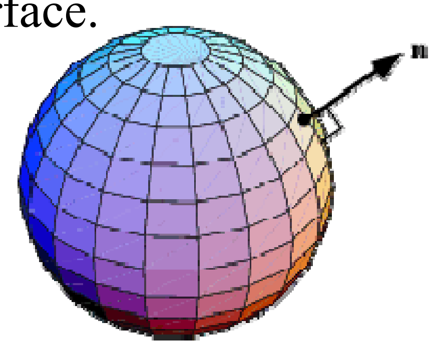
Local flatness

'Zoom' in on a curved surface and it looks flat.



Closed surfaces

Normal vectors point outward from a closed surface.

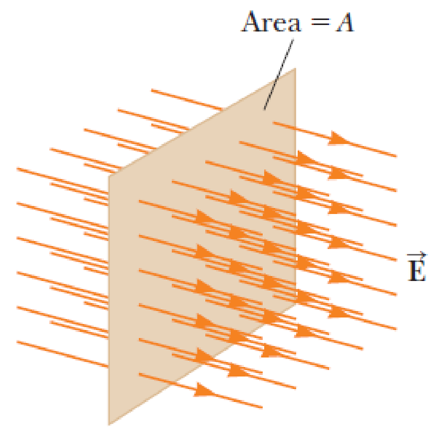


Electric flux, Φ

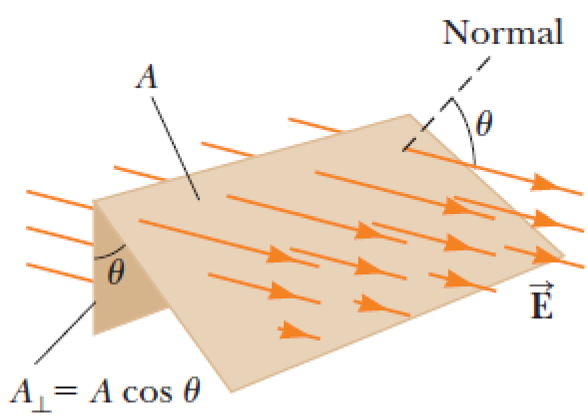
When the electric field is normal to a planar surface, the **electric flux** is the product of the magnitude of the electric field, *E*, and surface area, *A*.

For the diagram to the right the flux is simply

$$\Phi = EA$$



Assuming the electric field is still uniform, what happens to the electric flux when we tilt the flat surface at an angle θ to the direction of the electric field?



$$\begin{aligned} \Phi &= \vec{E}\cdot\vec{A} \\ &= EA\cos\theta \end{aligned}$$

(uniform field; flat surface at an angle)

Electric flux, Φ (cont.)

In general, the (a) electric field is <u>not always uniform</u> over the surface, and (b) the surface is <u>not always flat</u>.

For an electric field vector (magnitude & direction) that varies over an arbitrary area, the electric flux may be written as

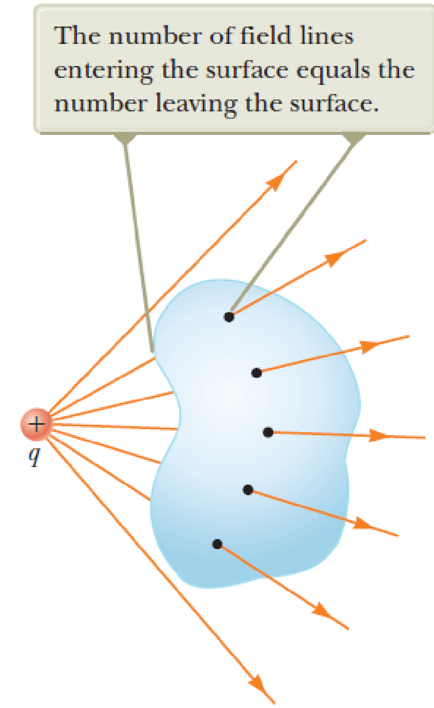
$$\Phi = \iint \vec{E} \cdot d\vec{A}$$

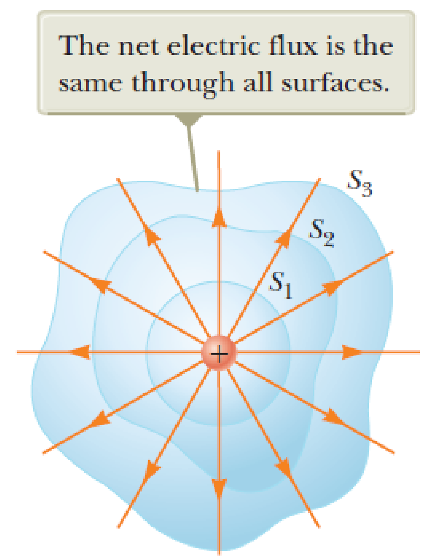
Consider a volume <u>enclosed</u> by an arbitrary surface. The flux through the closed surface may be written as a *closed surface integral*,

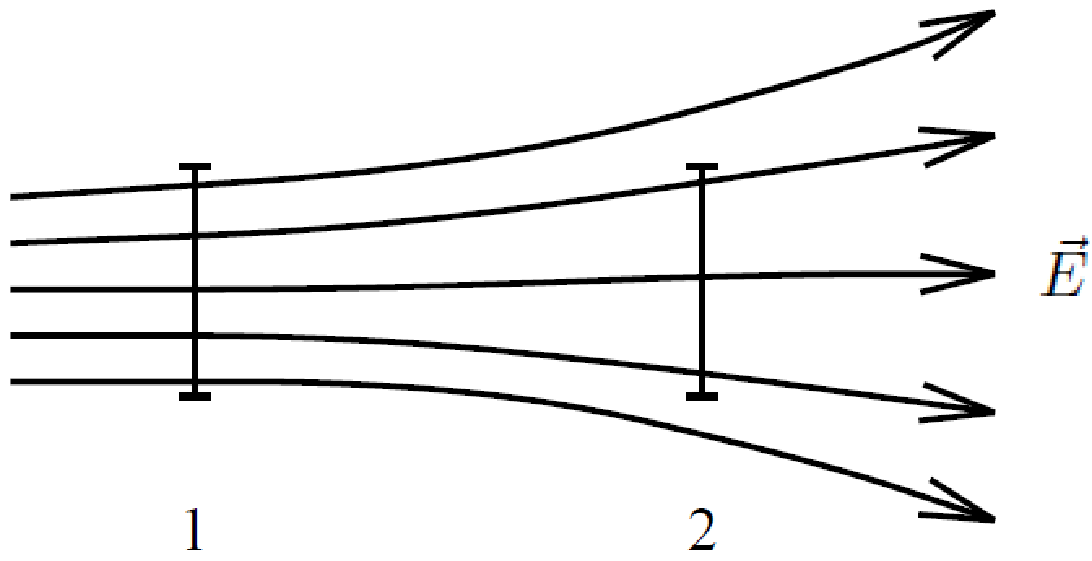
$$\Phi = \oiint \vec{E} \cdot d\vec{A}$$

The **net flux** through a closed surface is proportional to the number of lines <u>leaving</u> the surface *minus* the number of lines <u>entering</u> the surface.

If there is no charge inside the closed surface, then the net flux is zero!







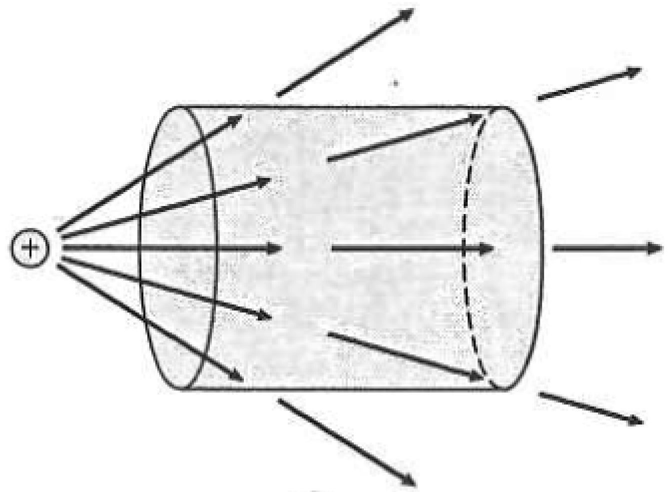
Which is the most true about the above diagram?

A. There is no flux through either surface.

B. The magnitude of the electric flux is greater through surface 1.

C. The magnitude of the electric flux is greater through surface 2.

D. The magnitude of the electric flux through both surfaces is equal and nonzero.

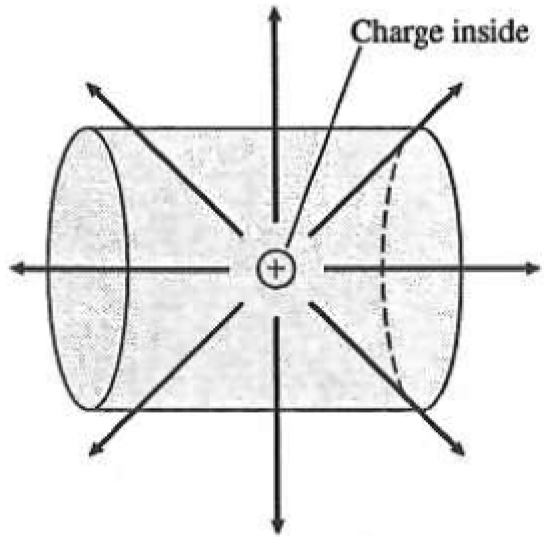


Which is true about the net flux through the above closed cylindrical surface?

A. The net flux is positive

B. The net flux is negative

C. The net flux is zero



Which is true about the net flux through the above closed cylindrical surface?

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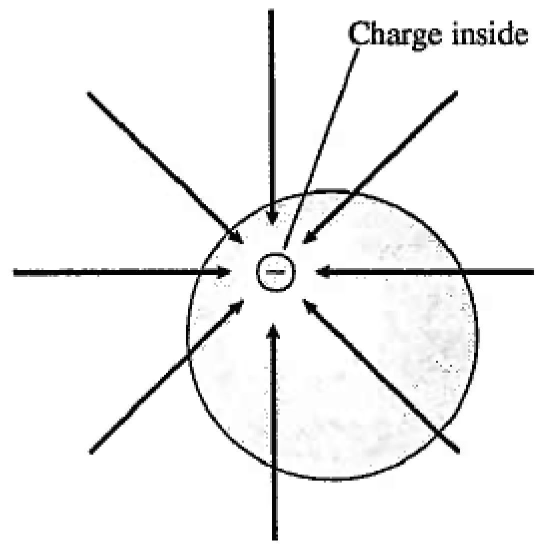
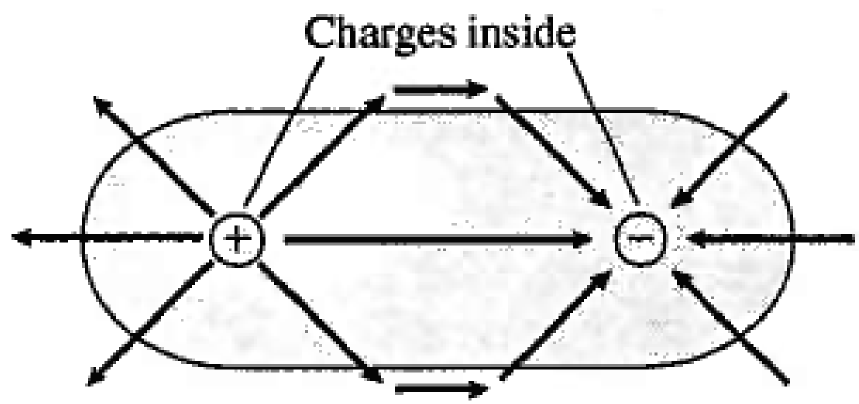


Which is true about the net flux through the above closed spherical surface?

A. The net flux is positive

B. The net flux is negative

C. The net flux is zero



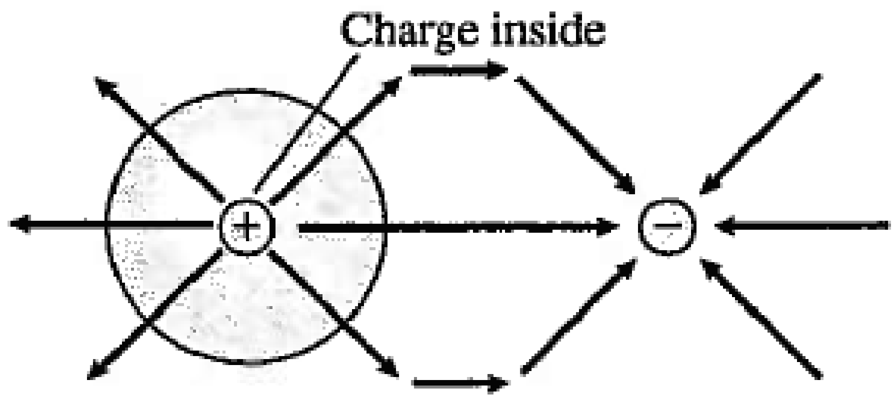
(the charges are equal in magnitude but opposite in sign)

Which is true about the net flux through the above closed surface?

A. The net flux is positive

B. The net flux is negative

C. The net flux is zero



Which is true about the net flux through the above closed spherical surface?

A. The net flux is positive

B. The net flux is negative

C. The net flux is zero

Gauss's law

The net flux of a closed surface is proportional to the charge inside the enclosure.

$$\Phi = \frac{q_{\text{in}}}{\epsilon_0}$$

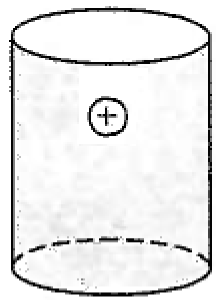
The electric permittivity, ϵ_0, is related to the Coulomb constant by the equation

$$k_e = \frac{1}{4\pi\epsilon_0} \quad \Longrightarrow \quad \epsilon_0 \approx 8.854 \times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2$$

Substituting our previous definition of flux into the left-hand-side of the equation at the top of this slide, we may write what is known as the integral form of Gauss's law.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

The closed cylindrical surface shown to the left has a positive charge of 177 pC inside the enclosure. The flux is Φ_{top} = 4.00 N·m²/C through the top side and Φ_{wall} = 13.0 N·m²/C through the wall. What is the flux through the bottom?



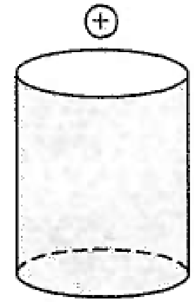
Solution

$$\Phi_{\mathbf{top}} + \Phi_{\mathbf{bottom}} + \Phi_{\mathbf{wall}} = \frac{q_{\mathbf{in}}}{\epsilon_0}$$

$$\Longrightarrow 4\ \text{N·m}^2/\text{C} + \Phi_{\text{bottom}} + 13\ \text{N·m}^2/\text{C}$$

$$= \frac{1.77\times10^{-10}\ \text{C}}{8.854\times10^{-12}\ \text{C}^2/\text{N·m}^2} \approx 20.0\ \text{N·m}^2/\text{C}$$

$$\Longrightarrow \boxed{\Phi_{\mathbf{bottom}} = 3.00\ \text{N·m}^2/\text{C}}$$



The closed cylindrical surface has a positive charge outside the enclosure. The flux is Φ_{top} = -15 N·m²/C through the top side and Φ_{bottom} = 5 N·m²/C. What is the flux through the wall?

A. 20 N·m²/C

B. 10 N·m²/C

C. -20 N·m²/C

D. -10 N·m²/C

<u>Solution</u>

$$\Phi_{\mathbf{top}} + \Phi_{\mathbf{bottom}} + \Phi_{\mathbf{wall}} = \frac{q_{\mathbf{in}}}{\epsilon_0}$$

⟹ $-15\ \text{N}\cdot\text{m}^2/\text{C} + 5\ \text{N}\cdot\text{m}^2/\text{C} + \Phi_{wall} = 0$

⟹ $\Phi_{wall} = 10\ \text{N}\cdot\text{m}^2/\text{C}$

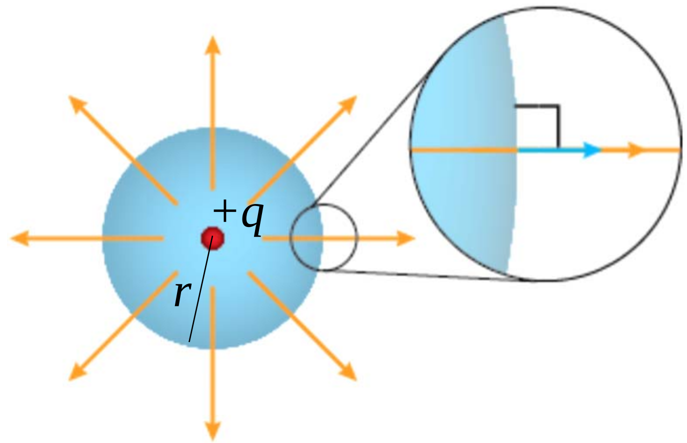
Finding the E-field using Gauss's Law

Most scenarios are general and there is no simple method to isolate the E-field.

Some special cases allow us to separate the E-field from the area integral.

<u>To separate the E-field:</u>
(1) The angle of incidence between the surface and the E-field must be the same everywhere at the surface.
(2) The magnitude of the E-field must be the same everywhere at the surface.

Find the E-field of a <u>positive point charge</u> using a <u>spherical surface</u>



$$\Phi = \oint\!\!\oint \vec{E} \cdot d\vec{A}$$

$$= \oint\!\!\oint E \cos(0°)\, dA$$

$$= E \oint\!\!\oint dA$$

$$= E\, r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= E\left(4\pi r^2\right)$$

We remember that the integral form of Gauss's law is

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

If the net charge enclosed is a point charge $+q$, then we may write

$$E\left(4\pi r^2\right) = \frac{q}{\epsilon_0}$$

Solving algebraically for the magnitude of the E-field gives

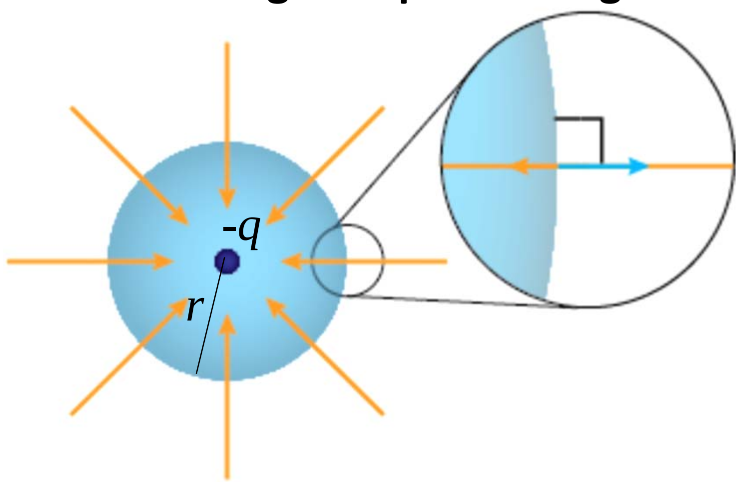
$$E = \frac{1}{4\pi r^2}\frac{q}{\epsilon_0}$$

Remembering (1) the relationship between the coulomb constant k_e and the electric permittivity ϵ_0 and (2) our 'radial only' symmetry,

$$\boxed{\vec{E} = k_e \frac{q}{r^2}\hat{r}}$$

What does this technique look like for a negative point charge?

Spherical surface centered at a negative point charge



$$\Phi = \oiint \vec{E} \cdot d\vec{A}$$
$$= \oiint E \cos(180^\circ)\, dA$$
$$= -E \oiint dA$$
$$= -E\, r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$
$$= -E\left(4\pi r^2\right)$$

Just as before, we write Gauss's law in integral form

$$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

If the net enclosed charge is a point charge -q, then we may write

$$-E\left(4\pi r^2\right) = -\frac{q}{\epsilon_0}$$

Solving algebraically for the magnitude of the E-field gives

$$E = \frac{1}{4\pi r^2}\frac{q}{\epsilon_0}$$

Again, remembering (1) the relationship between the coulomb constant, k_e, and (2) our initial field line direction, we find

$$\boxed{\vec{E} = -k_e \frac{q}{r^2}\hat{r}}$$

There is an empty spherical surface in space. Which of the following is always true?

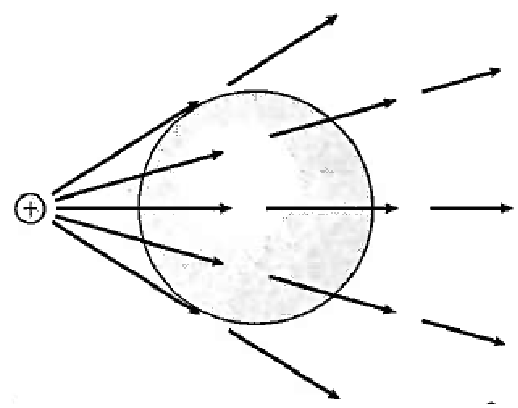
A. The electric field at the surface is zero

B. The flux through the surface is zero

C. Both the electric field at the surface and flux through the surface are zero.

D. Don't touch the surface or it will shock you!

(A) Is false because a charge outside could cause the field to be nonzero. This makes (C) false. (B) is the only statement that is always true.



Gauss's law and charged insulators

Charge density

The charge density, ρ, is defined as the amount of charge, q, over a volume, V.

For a homogenous distribution (the total charge, Q, is evenly spread over the volume), then the charge density is simply

$$\rho = \frac{Q}{V}$$

If the charge is not spread evenly throughout the volume (inhomogeneous), then the above expression is called the *average charge density*.

To find the charge density at a location in a non-uniformly distributed body, we look at a small amount of charge over a small a volume.

$$\rho = \frac{dq}{dV}$$

By knowing the density distribution, we can solve for the charge,

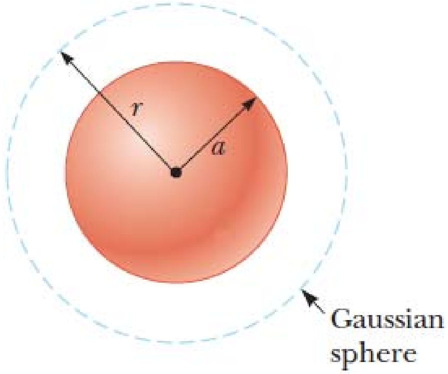
$$\int_0^Q dq = \iiint_{\mathcal{V}} \rho \, dV$$

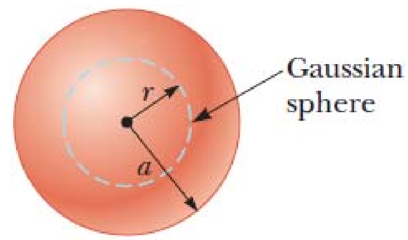
Thus, the charge may be written as the volume integral of the charge density

$$Q = \iiint_{\mathcal{V}} \rho \, dV$$

Example: insulating sphere with a radial charge distribution

Strategy: separately solve the regions outside and inside the charged sphere of radius *a*.





The charge per unit volume is radially dependent and given by $\rho(r) = 4Br$, where *B* is a positive constant.

Outside region:

Draw a Gaussian sphere centered with the spherical charge. The Gaussian sphere can extend from the radius of the charged sphere, *a*, out to infinity. We start from Gauss's law, where

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

Because we are calculating a sphere, the left-hand-side of the equation will be an angular surface integral. The right-hand-side will be a volume integral. The density only depends on the radius so we can separate the angular integrals from the radial integral giving,

$$E\,r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{1}{\epsilon_0} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^{a} \rho(r)\, r^2\, dr$$

Outside region (continued)

Again, the angular integrals are simply 4π.

$$E\,r^2\,(4\pi) = \frac{1}{\epsilon_0}\,(4\pi)\int_0^a \rho\,(r)\;r^2\,dr$$

Solving for the magnitude of the E-field outside gives,

$$E = \frac{1}{\epsilon_0 r^2}\int_0^a \rho\,(r)\;r^2\,dr$$

This equation is general and may be used for any radial charge distribution (and noting the direction of the E-field). Next, take our given function of charge density into the

$$E = \frac{1}{\epsilon_0 r^2}\int_0^a 4Br^3\,dr$$

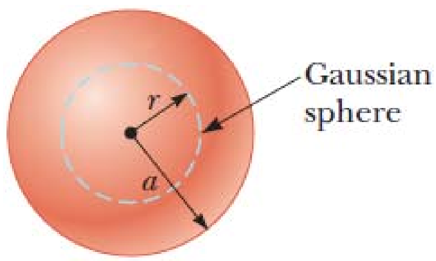
We know the integral of a power law, and taking the limits from 0 to *a* gives,

$$E = \frac{Ba^4}{\epsilon_0 r^2}$$

Because be is a positive constant and the charge density integral gave a positive value, then the net charge inside the Gaussian sphere is positive. Thus, the direction of the field is away from the sphere.

$$\vec{E} = \frac{Ba^4}{\epsilon_0 r^2}\hat{r} \quad \text{for} \quad r > a$$

Inside region:



Again, we draw a Gaussian sphere centered with the spherical charge. This time, the Gaussian sphere is inside the sphere and can extend from 0 to the radius, *a*.

We again start from our general equation of Gauss's law,

$$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

Just like the solution to the outside region, we take separate the angular integrals and the radial integral. There is only a some of the total charge enclosed by the Gaussian sphere.

Thus, we take the density integral all the way out to the surface of the Gaussian sphere.

$$E\,r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{1}{\epsilon_0} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^{r} \rho\,(r')\; r'^{\,2}\, dr'$$

Again, the angular integrals equate to a total of 4π.

$$E\,r^2\,(4\pi) = \frac{1}{\epsilon_0}\,(4\pi) \int_0^{r} \rho\,(r')\; r'^{\,2}\, dr'$$

<u>Inside region</u> (continued)

Solving for the magnitude of the E-field outside gives,

$$E = \frac{1}{\epsilon_0 r^2} \int_0^r \rho(r')\, r'^2\, dr'$$

This equation is general and may be used for any radial charge distribution inside the sphere. Next, take our given function of charge density into the

$$E = \frac{1}{\epsilon_0 r^2} \int_0^r 4Br'^3\, dr'$$

We know the integral of a power law, and taking the limits from 0 to *a* gives,

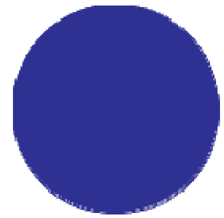
$$E = \frac{Br^4}{\epsilon_0 r^2}$$

Again, the direction of the field is away from the sphere. After simplifying,

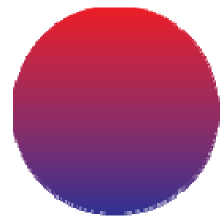
$$\boxed{\vec{E} = \frac{Br^2}{\epsilon_0}\hat{r} \quad \text{for} \quad r < a}$$

Which spherical charge distribution can not be solved using Gauss's law and drawing a spherical shell? (positive, negative)

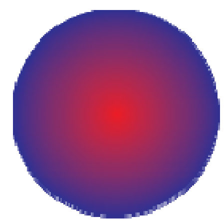
A. A negative uniform charge distribution.



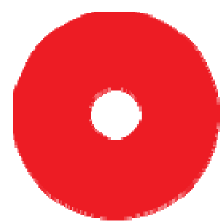
B. A charge distribution that begins positive at the top of the sphere and becomes negative at the bottom.



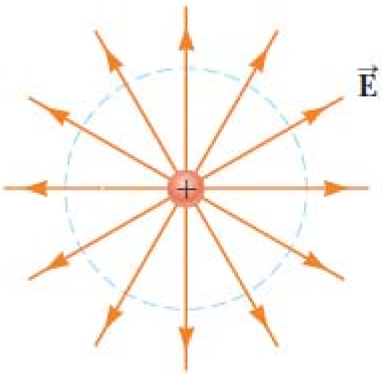
C. A radial distribution that begins positive at the center and becomes negative at the surface.



D. A positive uniform density with a small sphere cut from the sphere and centered at the origin.



Example: infinitely long straight line of charge



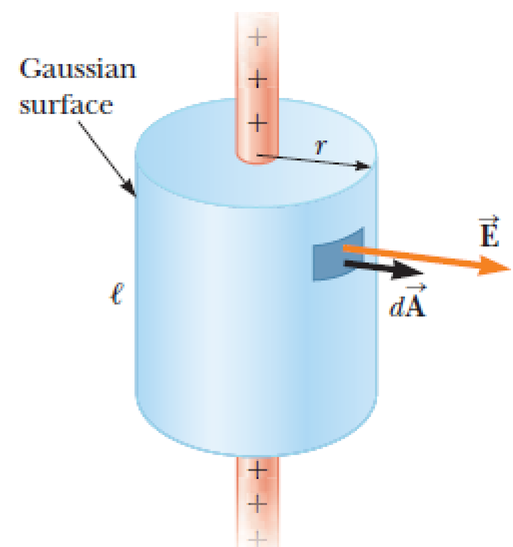
The infinite line of charge is a straight string of charge with a linear charge density, λ. The linear charge density is defined by

$$\lambda = \frac{dq}{d\ell} \quad \Longrightarrow \quad Q = \int \lambda \, d\ell$$

The 'end view' on the left shows how the field lines spread out from a positive source (point in to a negative source). An 'end view' of an infinite cylindrical Gaussian surface centered at the line charge is perpendicular to the field lines. The Gaussian cylinder also stretches out to infinity and thereby always encloses the line charge.

We begin solving for the E-field by writing our general equation,

$$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$



The Gaussian surface shown to the right shows the area is a function of the angle, ϕ, and the height, z. The left-hand-side of the equation has the relationship

$$\begin{aligned} \oiint \vec{E} \cdot d\vec{A} &= E \int_0^{2\pi} r \, d\phi \int_{-\infty}^{\infty} dz \\ &= E\,(2\pi r) \int_{-\infty}^{\infty} dz \end{aligned}$$

Infinitely long straight line of charge (continued)

The right-hand-side can be evaluated using the definition of the linear charge density.

$$\frac{q_{\mathrm{in}}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_{-\infty}^{\infty} \lambda\, dz$$

If the linear charge density is constant (does not depend on the height, z), then we can pull the density out of the integral.

$$\frac{q_{\mathrm{in}}}{\epsilon_0} = \frac{\lambda}{\epsilon_0}\int_{-\infty}^{\infty} dz$$

Combining the left- and right-hand-sides together gives

$$2\pi r E\left[\int_{-\infty}^{\infty} dz\right] = \frac{\lambda}{\epsilon_0}\left[\int_{-\infty}^{\infty} dz\right]$$

Really big number!

Same big number

From algebra we can divide both sides by the same big number. Therefore,

$$2\pi r E = \frac{\lambda}{\epsilon_0}$$

$$\Longrightarrow \boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}}$$

Example: Infinitely long cylinder with a radial charge distribution

Instead of a line charge, let us assume an infinite cylinder extending in the *z* direction with a radius, *a*. Let's assume a uniform charge density, ρ_0 (although any radially dependent charge density works).

Outside region:

As always, we begin with the general description of Gauss's law.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

As we showed during the last example, the area is the integral with respect to the angle and the height. The right side is now a volume integral in cylindrical coordinates. Thus, the above equation becomes

$$Er \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz = \frac{\rho_0}{\epsilon_0} \int_0^{2\pi} d\phi \int_0^a r\, dr \int_{-\infty}^{\infty} dz$$

Just like last time on the left-hand-side, the angular integral reduces to 2π and the other integral is a big number. The right-hand-side is similar but there is also a simple radial integral. We can reduce the equation to

$$(2\pi r)\, E \left[\int_{-\infty}^{\infty} dz\right] = \frac{\pi a^2 \rho_0}{\epsilon_0} \left[\int_{-\infty}^{\infty} dz\right]$$

Cancelling the 'really large' numbers and doing a little algebra, we find

$$E = \frac{a^2 \rho_0}{2\epsilon_0 r}$$

The field lines are only in the radial direction in cylindrical coordinates. Therefore,

$$\boxed{\vec{E} = \frac{a^2 \rho_0}{2\epsilon_0 r} \hat{r}}$$

Inside region:

Once again, we begin with the general description of Gauss's law.

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

We explicitly write the separable integrals like the outside case, except the radial limit on the density integral is at the edge of the Gaussian surface.

$$Er \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz = \frac{\rho_0}{\epsilon_0} \int_0^{2\pi} d\phi \int_0^r r'\, dr' \int_{-\infty}^{\infty} dz$$

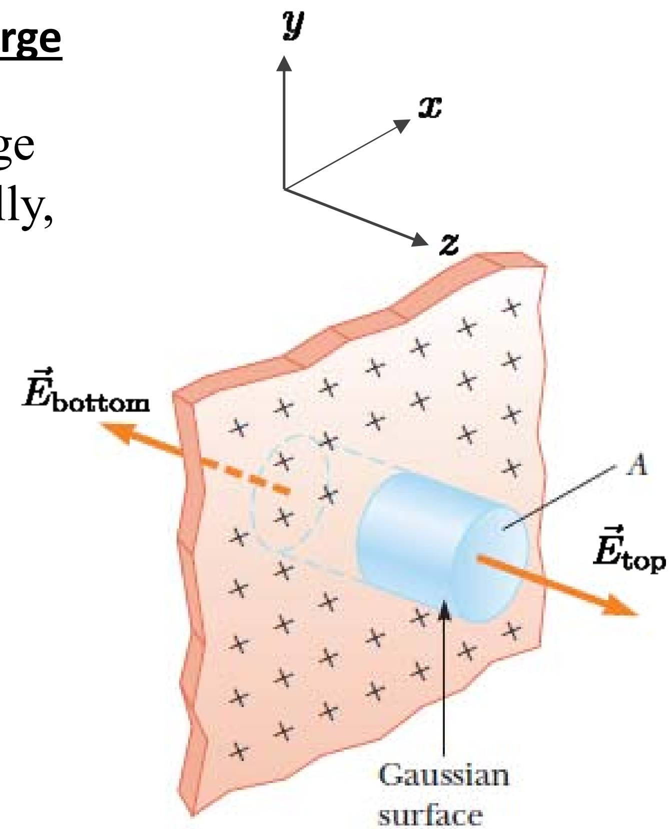
Evaluating the non-infinite integrals gives

$$(2\pi r)\, E \left[\int_{-\infty}^{\infty} dz\right] = \frac{\pi r^2 \rho_0}{\epsilon_0} \left[\int_{-\infty}^{\infty} dz\right]$$

Solving algebraically and adding the radial direction information of the E-field gives

$$\boxed{\vec{E} = \frac{r \rho_0}{2\epsilon_0} \hat{r}}$$

Example: infinite sheet of charge



Now consider a planar sheet of charge with constant charge density stretching out to infinity in two directions. Basically, an electrically charged thin crust pizza where the radius expands to infinity.

We define the surface density as

$$\sigma = \frac{dq}{dA}$$

The total charge is given as

$$Q = \iint_A \sigma \, dA$$

We shall begin with the general definition of Gauss's law in integral form

$$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

Note how the Gaussian surface is drawn in the diagram. There are two sides to the cylinder and the wall stretches out to infinity. Thus, both the top and bottom parts of the cylinder are required to enclose the cylinder.

$$\oiint \vec{E} \cdot d\vec{A} = E_{\text{top}} \int_0^{2\pi} d\phi \int_0^{\infty} r\, dr + E_{\text{bottom}} \int_0^{2\pi} d\phi \int_0^{\infty} r\, dr$$

infinite sheet of charge (continued)

The top and bottom electric fields must be equal in magnitude. Thus, the left-hand-side reduces to

$$\oiint \vec{E} \cdot d\vec{A} = 2E \int_0^{2\pi} d\phi \int_0^{\infty} r\, dr$$

Note that the radial integral is a REALLY BIG number.

The right-hand-side of Gauss's law in integral can also be evaluated from the definition of the surface charge density.

$$\frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma}{\epsilon_0} \int_0^{2\pi} d\phi \int_0^{\infty} r\, dr$$

We can now rewrite Gauss's law with the expanded expressions above.

$$2E \left[\int_0^{2\pi} d\phi \int_0^{\infty} r\, dr \right] = \frac{\sigma}{\epsilon_0} \left[\int_0^{2\pi} d\phi \int_0^{\infty} r\, dr \right]$$

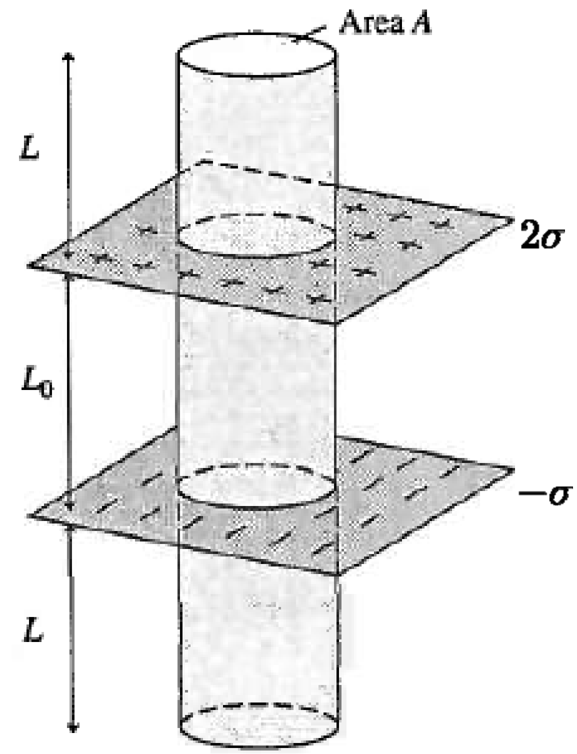
Canceling the big numbers on both sides gives the magnitude of the E-field through one side is given by

$$E = \frac{\sigma}{2\epsilon_0}$$

The field above the plane points in the positive direction and the field below the plane points in the negative direction. Thus,

$$\boxed{\vec{E}_{top} = \frac{\sigma}{2\epsilon_0}\hat{z} \quad \text{and} \quad \vec{E}_{bottom} = -\frac{\sigma}{2\epsilon_0}\hat{z}}$$

Two infinite sheets are parallel with each other. Which of the following statements is true?



A. The electric field in between the sheets is *zero*.

B. The magnitude of the E-field above both sheets is *greater* than the magnitude of the E-field below both sheets.

C. The magnitude of the E-field above both sheets is *equal* to the magnitude of the E-field below both sheets.

D. The E-field between the sheets points upward

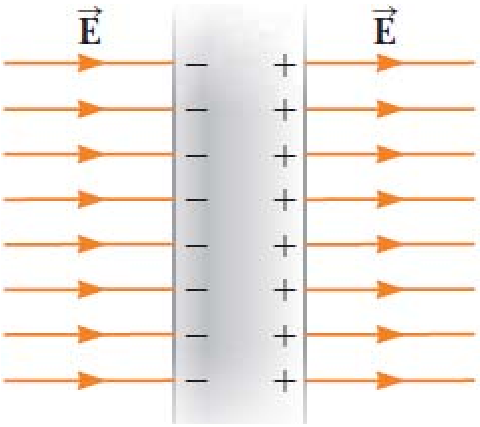
Conductors

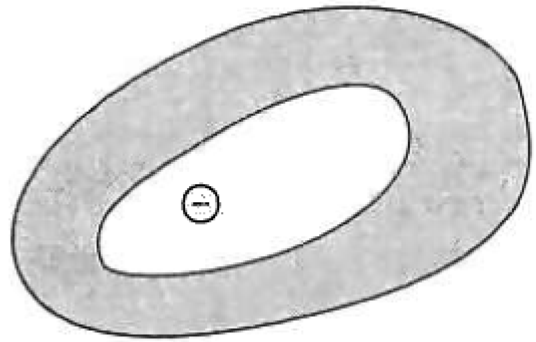
A conductor is in **electrostatic equilibrium** when there is no net motion inside a conductor.

A conductor in electrostatic equilibrium has the following four properties:

1. The electric field is zero everywhere inside the conductor, whether the conductor is solid or hollow.
2. If the conductor is isolated and carries a charge, the charge resides on its surface.
3. The electric field at a point just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude σ/ϵ_0, where σ is the surface charge density at that point.
4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.

In a conductor, the electrons move towards and external field. This cancels the field inside the conductor.





There is a hole inside a conductor, and the conductor has a charge of +10 nC. There is a point charge of -10 nC inside the void of the conductor. What is the charge on the outside surface of the conductor?

A. 0

B. +5

C. -5

D. +10