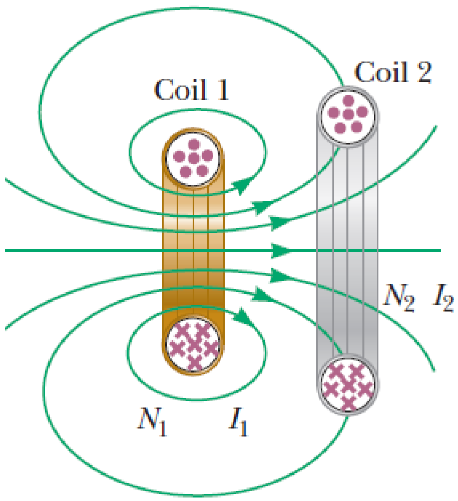
Chapter 30: Inductance

- Mutual inductance
- Self inductance and inductors
- Magnetic field energy
- Resistor-inductor circuit
- Inductor-capacitor circuit
- Inductor-resistor-capacitor series circuit

Mutual inductance

Mutual inductance occurs when there is a time changing current in one circuit, which induces a time changing current in a neighboring circuit.



Consider two coils as shown; a time changing current in coil 1 caused a time changing magnetic field. This magnetic field passes through the center of coil 2.

The time changing magnetic flux in the center of coil 2 causes an emf,

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{B,2}}{dt}$$

We know that the magnetic flux through the center of coil 2 is proportional to the current in coil 1, $N_2 \Phi_{B,2} = M_{21} I_1$. Therefore, we may rewrite the emf as

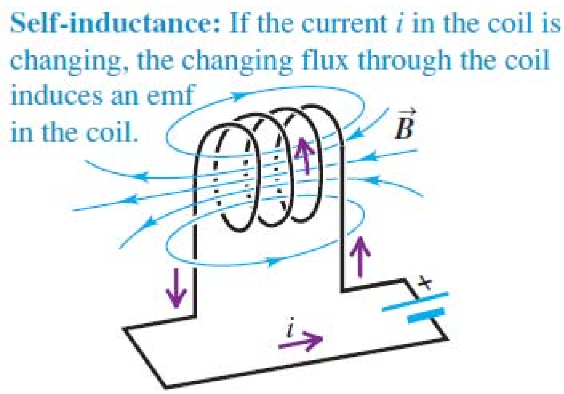
$$\mathcal{E}_2 = -M_{21} \frac{dI_1}{dt}$$

In mutual induction, the emf induced in a coil is always proportional to the rate of current change in the other coil. It turns out that $M_{21} = M_{12} = M$. Thus,

$$\mathcal{E}_1 = -M \frac{dI_2}{dt} \quad \text{and} \quad \mathcal{E}_2 = -M \frac{dI_1}{dt}$$

Self-inductance

Any circuit that carries a varying current has an emf induced in it by the variation in its own magnetic field, which is known as the self-induced emf.



We define the (self) inductance of a coil as $L = N\dfrac{\Phi_B}{I}$.

Rearranging the inductance equation and taking the time derivative gives

$$N\frac{d\Phi_B}{dt} = L\frac{dI}{dt}$$

From Faraday's law for a coil with turns, the self-induced emf is

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

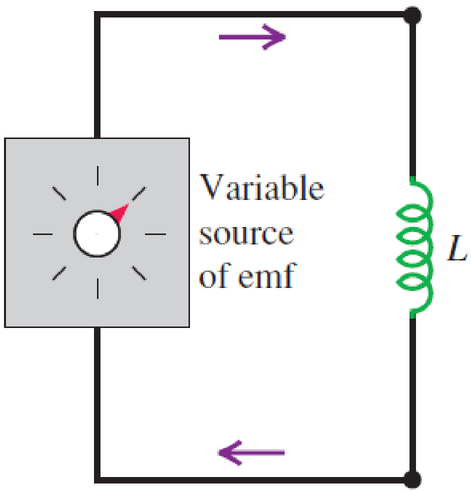
A technician wraps wire around a tube of length $\ell = 35.0$ cm having a diameter of d = 7.50 cm. When the windings are evenly spread over the full length of the tube, the result is a solenoid with 600 turns of wire (where we can make the ideal solenoid approximation).

(a) What is the inductance of the solenoid?
(b) If the current in this solenoid increases at a rate of 4.50 A/s, then what is the self-inductance emf in the solenoid?

Inductors

A coiled wire in a circuit is called an inductor. The usual circuit symbol for an inductor is a short looping wire,  .

Let us call the conservative electric field from charges in a circuit $\vec{E}_c$, and the magnetically induced nonconservative electric field $\vec{E}_n$.



Consider the circuit shown to the left. The line integral of around the circuit is the negative of the rate of change of flux through the circuit.

$$\oint \vec{E}_n \cdot d\vec{\ell} = -N\frac{d\Phi_b}{dt} = -L\frac{dI}{dt}$$

$\vec{E}_n$ is zero except at the inductor, so we only need to consider the open line integral $\int \vec{E}_n \cdot d\vec{\ell} = -L\frac{dI}{dt}$.

The line integral of $\vec{E}_c$ is just the potential across the inductor, $\int \vec{E}_c \cdot d\vec{\ell} = V$.

Also, we know that $\vec{E}_n + \vec{E}_c = 0$. Therefore, the potential difference across the inductor is

$$V = L\frac{dI}{dt}$$

At what rate would the current in a 100 mH inductor have to induce an emf of 1000 V in the inductor?

A. 10 A/s

B. 100 A/s

C. 1000 A/s

D. 10,000 A/s

An inductor has a current $I(t) = (0.510\ \text{A}) \cos[(250\ \text{rad/s})t]$ flowing through it. If the maximum emf across the inductor is equal to 0.600 V, what is the self-inductance of the inductor?

A. 3.76 mH

B. 4.71 mH

C. 5.65 mH

D. 6.59 mH

Which of the following statements about inductors is correct?

A. Inductors store energy by building up charge.

B. When an inductor and a resistor are connected in series with a DC battery, the current in the circuit is reduced to zero in one time constant.

C. An inductor always resists any change in the current through it.

D. When an inductor and a resistor are connected in series with a DC battery, the current in the circuit is zero after a very long time.

Magnetic field energy

Energy is stored in an inductor that is carrying a current.

Consider a circuit starting with $I = 0$, and with an increasing current $dI/dt > 0$ for $t > 0$. The power (rate energy is being delivered to the inductor) is given by

$$P = IV = IL\frac{dI}{dt}$$

By definition, the power is the change in energy per change in time. Therefore we may write

$$dU = P\,dt = IL\,dI$$

Therefore, the energy stored in an inductor from a steady current I, follows as

$$U = \frac{1}{2}LI^2$$

Similar to to the electric field energy density in capacitors, we may also write the magnetic field energy density for the magnetic field in the inductor as

$$u = \frac{B^2}{2\mu}$$ (μ is the magnetic permeability, $\mu = \mu_0 K_m$)

where K_m is the relative magnetic permeability.

What is the energy density in the magnetic field 25 cm from a long straight wire carrying a current of 12 A? ($\mu 0 = 4\pi \times 10^{-7}$ T m/A)

A. 3.7×10^{-5} J/m^3

B. 7.3×10^{-5} J/m^3

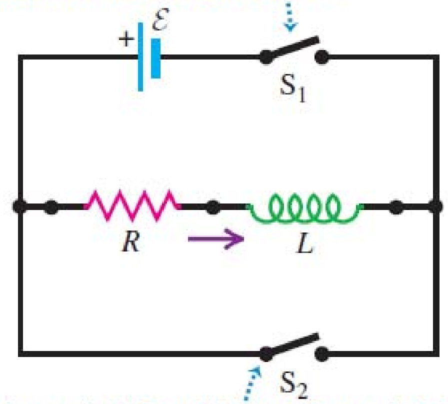
C. 1.2×10^{-4} J/m^3

D. 3.6×10^{-4} J/m^3

The resistor-inductor circuit (growth)

A circuit that includes both a resistor and an inductor, and possibly a source of emf, is called an ***RL* circuit.**

Consider the circuit to the right. Suppose both switches are open to begin with, and then at some initial time $t = 0$ we close switch S_1.



The voltage drop across the resistor is $V_R = IR$ and the voltage drop across the inductor is $V_L = L\dfrac{dI}{dt}$.

Kirchoff's loop law around the closed circuit gives $\mathcal{E} - IR - L\dfrac{dI}{dt} = 0$.

$$\Longrightarrow \quad \frac{dI}{dt} = \frac{\mathcal{E}}{L} - \frac{R}{L}I$$

RL time constant $\tau = L/R$

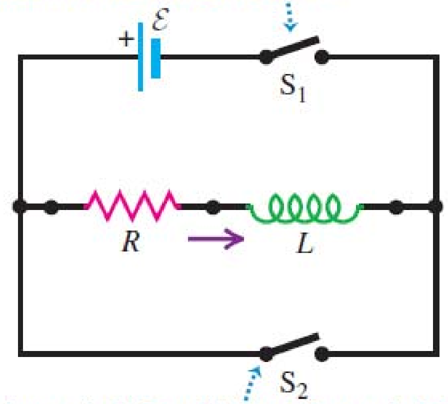
Solving this equation for the current I gives $I(t) = \dfrac{\mathcal{E}}{R}\left[1 - e^{-(R/L)t}\right]$.

The power delivered from the source to the RL circuit is

$$P = I\mathcal{E} = I^2 R + IL\frac{dI}{dt}$$

The resistor-inductor circuit (decay)

Suppose S_1 switch has been closed for a long time with steady state current I_0. Then we close S_2 which bypasses the battery and quickly open S_1 to keep from shorting the battery for a long time period.



Kirchoff's loop law around the closed circuit gives

$$IR - L\frac{dI}{dt} = 0$$

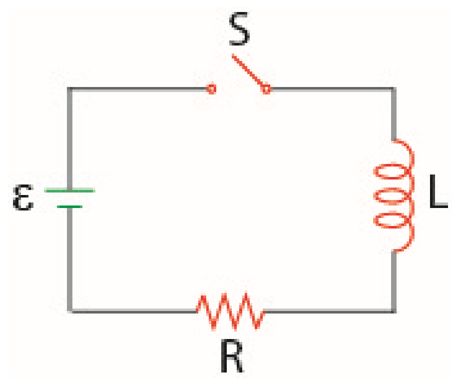
$$\Longrightarrow \quad \frac{dI}{dt} = \frac{R}{L}I$$

Solving for the current *I* in decaying from the initial current $I_0 = \mathcal{E}/R$, we get

$$I(t) = I_0 e^{-(R/L)t}$$

There is no power delivered from a source this time, and therefore

$$0 = I^2R + IL\frac{dI}{dt}$$



Consider the circuit in the diagram, where $\mathcal{E} = 12.0$ V, $L = 8.00$ mH, and $R = 5.00\ \Omega$.

(a) What is the inductive time constant of the circuit?

(b) What is the current in the circuit 250 μs after the switch is closed?

(c) After what time interval does the current reach 75% of its maximum value?

(d) What is the value of the final steady-state current?

A resistor and an ideal inductor are connected in series to an ideal battery having a constant terminal voltage V_0. At the moment contact is made with the battery, the voltage across the resistor is

A. V_0/e.

B. V_0.

C. $V_0/2$.

D. zero.

A resistor and an ideal inductor are connected in series to an ideal battery having a constant terminal voltage V_0. At the moment contact is made with the battery, the voltage across the inductor is

A. V_0/e.

B. V_0.

C. V_0/2.

D. zero.

A series LR circuit consists of a 2.0 H inductor with negligible internal resistance, a 100 Ω resistor, an open switch, and a 9.0 V ideal power source. After the switch is closed, what is the maximum power delivered by the power supply?

A. 0.090 W

B. 0.40 W

C. 0.81 W

D. 1.43 W

A resistor, an inductor, and a switch are all connected in series to an ideal battery of constant terminal voltage. What does the time constant for the circuit represent?

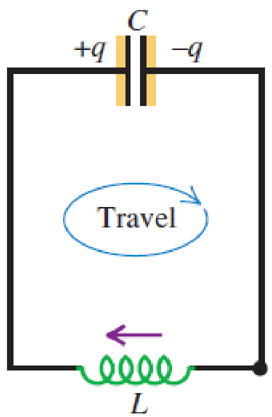
A. The time constant represents the time required for the current to reach 25% of the maximum current.

B. The time constant represents the time required for the current to reach 37% of the maximum current.

C. The time constant represents the time required for the current to reach 63% of the maximum current.

D. The time constant represents the time required for the current to reach 75% of the maximum current.

Conservative inductor-capacitor circuit



An LC circuit has a different mode of operation than the growth/decay operation modes we saw previously with RC and RL circuits.

Kirchoff's loop rule gives $-L\frac{dI}{dt} - \frac{q}{C} = 0.$

We can substitute $I = dq/dt$ into the above equation to get $\frac{d^2q}{dt^2} = -\frac{q}{LC}.$

This is of the same form as the harmonic oscillator equation $\frac{d^2x}{dt^2} = -\frac{k}{m}x.$

The solution of the charge on a capacitor is of the same form, which is

$$q(t) = Q_{\max}\cos(\omega t + \phi)$$

where the angular frequency is $\omega = 1/\sqrt{LC}$, ϕ is a phase constant, and Q_{max} is the charge constant.

Taking the time derivative $I = \frac{dq}{dt}$ gives the current as a function of time,

$$I(t) = -\omega Q_{\max}\sin(\omega t + \phi)$$

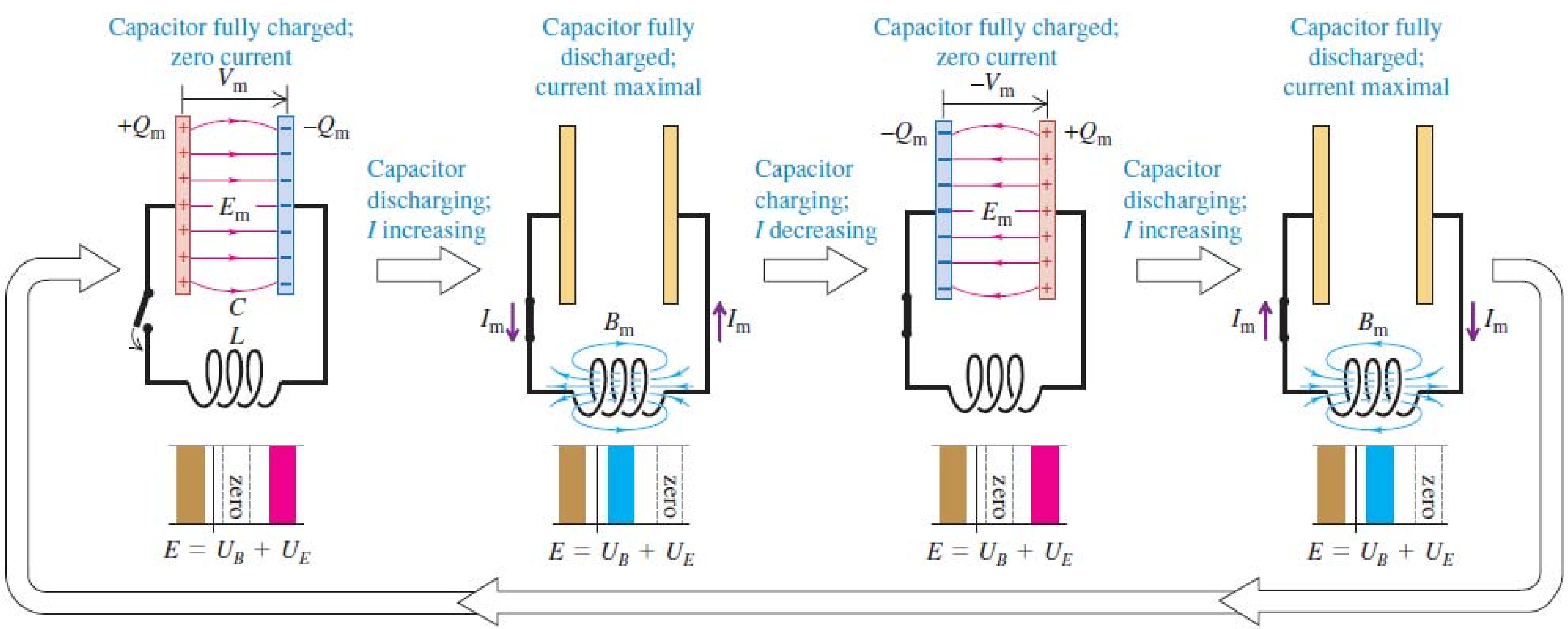
Energy conservation in a LC circuit

An ideal LC circuit is a conservative system.

When the capacitor is charged to its maximum value Q_{max}, there is no energy being stored in its conductor. Thus, the total amount of energy in the system is given by $Q/2C$. At any given time we have the energy,

$$\frac{1}{2}LI^2 + \frac{q}{2C} = \frac{Q_{max}}{2C}$$

Solving the equation for the current gives $I = \dfrac{\sqrt{Q_{max}^2 - q^2}}{\sqrt{LC}}$.



In an LC circuit containing a 40 mH ideal inductor and a 1.2 mF capacitor, the maximum charge on the capacitor is 45 mC during the oscillations. What is the maximum current through the inductor during the oscillations?

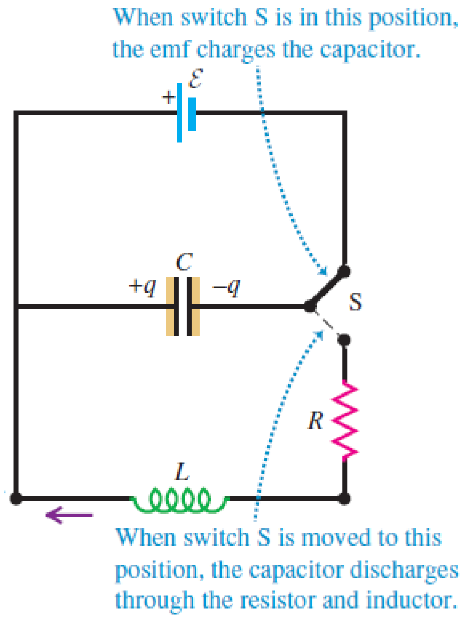
A. 2.5 A

B. 3.7 A

C. 6.5 A

D. 9.8 A

Inductor-resistor-capacitor circuit



An LRC circuit can oscillate just like the LC circuit, except that the system is no longer conservative due to losses from the presence of the resistor.

After the capacitor is charged and the switch to the LRC circuit is closed, we may use Kirchoff's loop rule to get

$$-IR - L\frac{dI}{dt} - \frac{q}{C} = 0$$

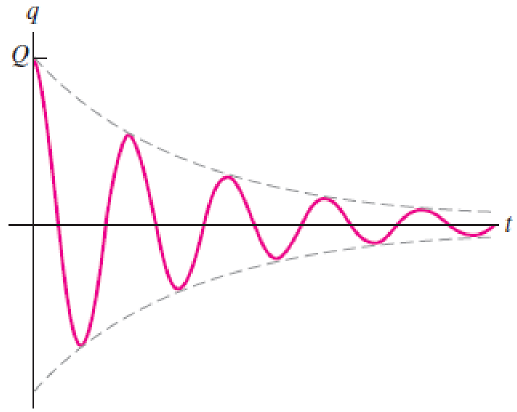
Substituting the expression $I = dq/dt$ into the above equation gives

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

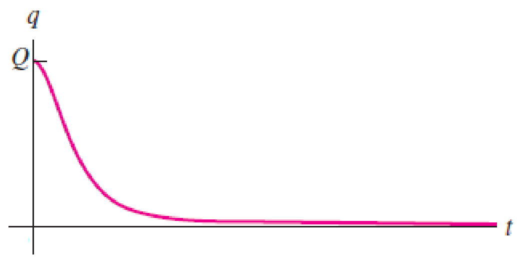
The solution of this equation is of the form $q(t) = Q_{\max} e^{-(R/L)t} \cos(\Omega t + \phi)$.

The angular frequency in the solution is $\Omega = \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}$.

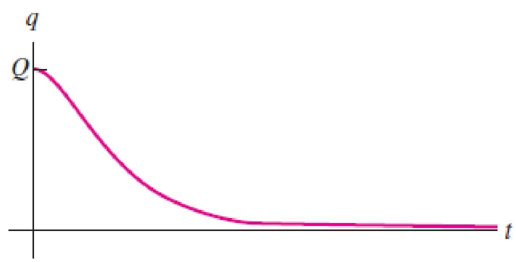
Damping in a LRC circuit



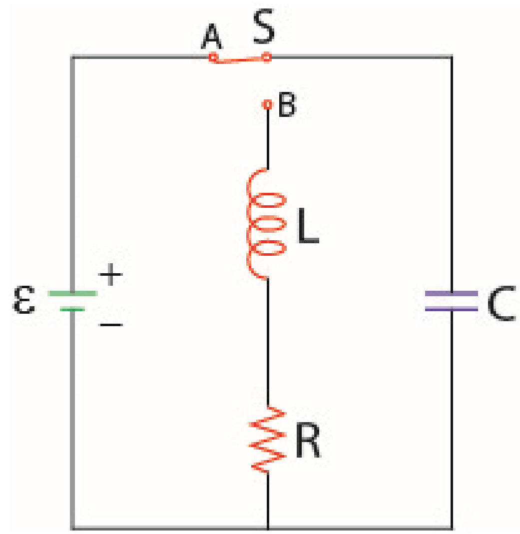
The system is underdamped when the resistance R is relatively small and the circuit still oscillates, but with damped harmonic motion.



The system is critically damped when the resistance R is increased to the point where the system is no longer allowed to oscillate.



The system is overdamped when the resistance R is increased beyond the critically damped value, and the charge on the capacitor takes longer to dissipate.



In the diagram, let $R = 3.50\ \Omega$, $L = 6.00$ mH, and C = 17.0 μF.

(a) Calculate the frequency of the damped oscillation of the circuit when the switch is thrown to position B.

(b) What is the critical resistance for damped oscillations?