Chapter 37: Relativity

- Invariance of physical laws
- Relativity and simultaneity
- Relativity of time intervals
- Relativity of length
- Lorentz transforms
- Doppler effect for electromagnetic waves
- Relativistic momentum
- Relativistic work and energy
- Newtonian mechanics and relativity

Postulates of special relativity

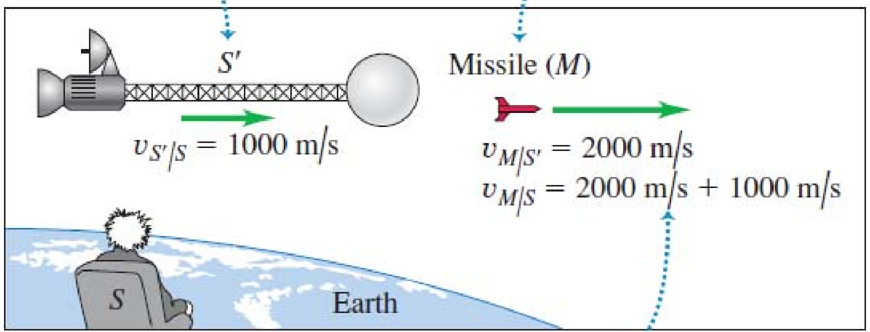
Einstein based his theory of special relativity on just two simple postulates:

1st postulate - The principle of relativity

The laws of physics are the same in every inertial frame of reference.

2nd postulate - The constancy of the speed of light

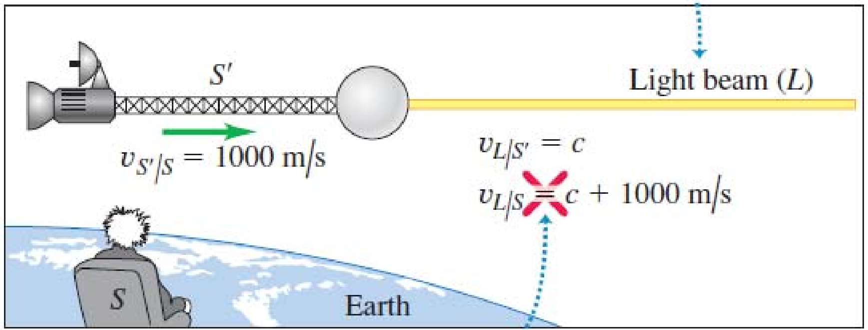
The laws of physics are the same in every inertial frame of reference.



A spaceship (S') moves with speed $v_{S'|S} = 1000$ m/s relative to an observer on earth (S).

A missile (M) is fired with speed $v_{M|S'} = 2000$ m/s relative to the spaceship.

NEWTONIAN MECHANICS HOLDS: Newtonian mechanics tells us correctly that the missile moves with speed $v_{M|S} = 3000$ m/s relative to the observer on earth.



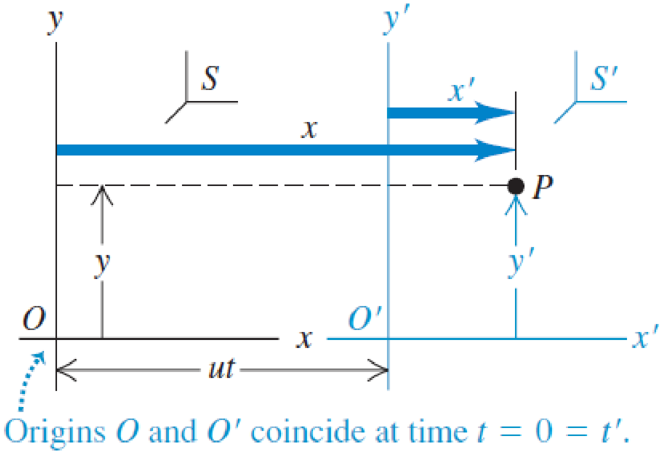
A light beam (L) is emitted from the spaceship at speed c.

NEWTONIAN MECHANICS FAILS: Newtonian mechanics tells us *incorrectly* that the light moves at a speed greater than c relative to the observer on earth ... which would contradict Einstein's second postulate.

A star is moving towards the earth with a speed at 90% the speed of light. It emits light, which moves away from the star at the speed of light. Relative to us on earth, what is the speed of the light moving toward us from the star?

A. 1.1 c
[✓ B. c]
C. 0.9 c
D. 0.1 c

Galilean coordinate transformation



Origins O and O' coincide at time $t = 0 = t'$.

Suppose we have two frames of reference, S described by the coordinates x and y, and S', described by the coordinates x' and y'.

The origin O' of frame S' moves relative to the origin O of frame S with constant velocity u along the common x (or x') axis.

Setting our clocks so that the two origins coincide at time $t = 0$, their separation at a later time t is ut.

The coordinates in frame S can be described in terms of the coordinate frame S',

$$x = x' + ut \qquad y = y' \qquad z = z'$$

Important note: the positive x-direction must be the direction in which the origin O moves relative to the origin O'. The diagram shows this direction being to the right. If instead O moves to the left relative to O' you must define the positive x-direction to be to the left.

Galilean velocity transformation

If a particle moves in the x-direction, its instantaneous velocity v_x as measured by an observer stationary in S is $v_x = dx/dt$.

The particle's velocity v_x' measured from in the S' frame is $v_x' = dx'/dt$.

The time derivative of the Galilean coordinate transformation, $x = x' + ut$, gives

$$\frac{dx}{dt} = \frac{dx'}{dt} + u$$

Thus, the Galilean velocity transformation for one-dimensional motion is

$$v_x = v_x' + u$$

There is a **<u>fundamental problem</u>** with the Galilean velocity transformation, which is when we look at the velocity of light,

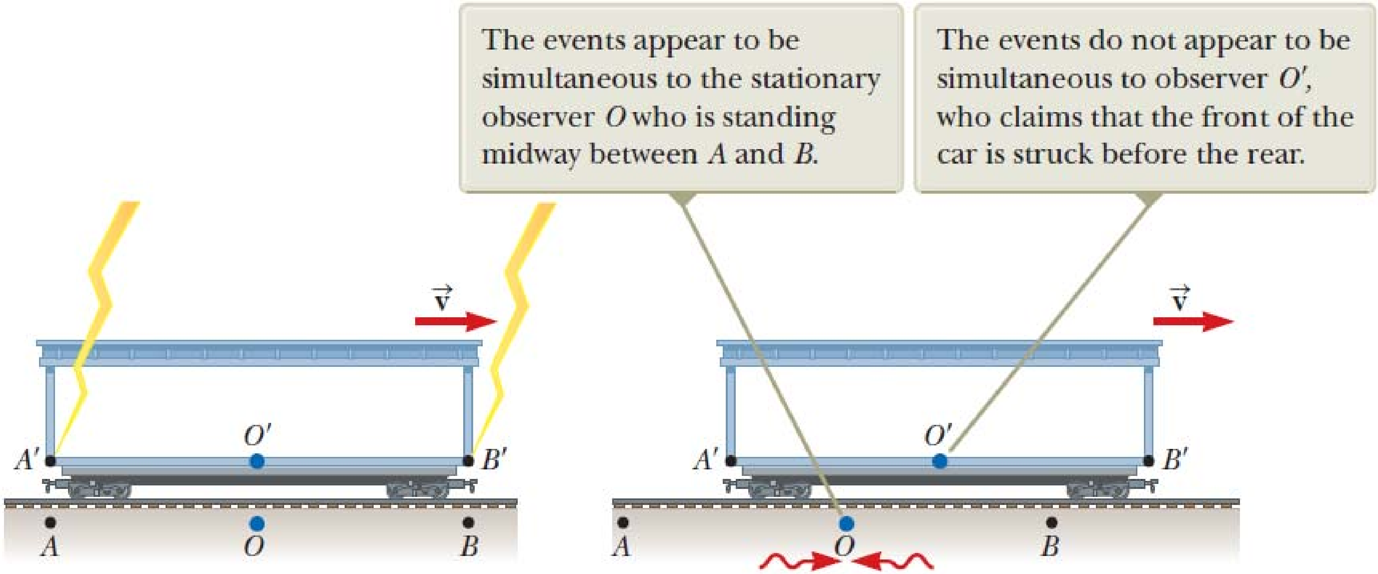
$$c = c' + u$$

which is **<u>not true</u>**, as we know that the speed of light is the same in all reference frames, $c = c'$.

Simultaneity

Two events that are simultaneous in one frame of reference are not simultaneous in a second frame that is moving relative to the first, even if both are inertial frames.

Example via Einstein thought experiment:



The events appear to be simultaneous to the stationary observer *O* who is standing midway between *A* and *B*.

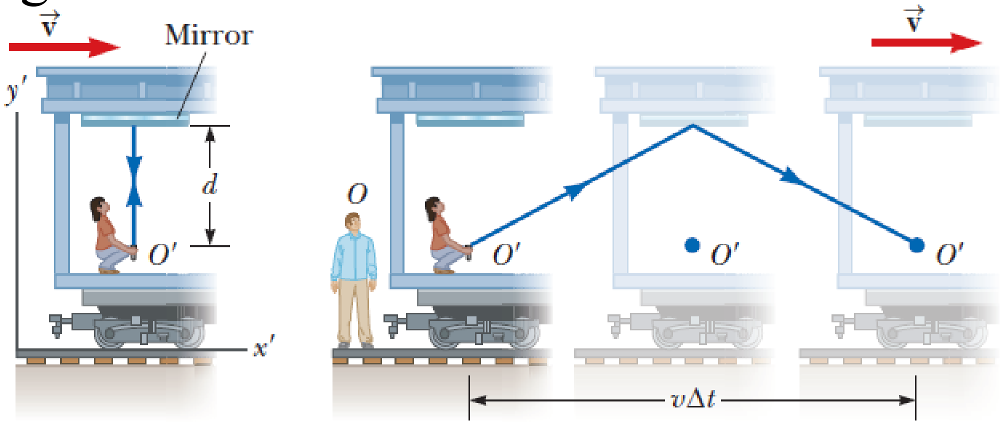
The events do not appear to be simultaneous to observer *O*′, who claims that the front of the car is struck before the rear.

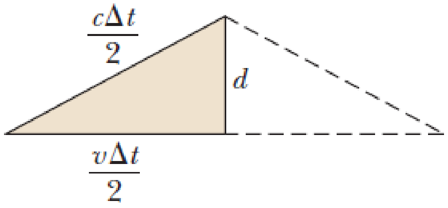
Two lightning bolts strike the ends of a moving boxcar.

The leftward-traveling light signal has already passed *O*′, but the rightward-traveling signal has not yet reached *O*′.

Relativity of time intervals

Consider a boxcar moving to the right with a speed *v*. A mirror is fixed to the ceiling, and observer *O′* at rest in the frame attached to the vehicle holds a flashlight a distance *d* below the mirror.





The time for the pulse to travel from *O′* to the mirror and back is $\Delta t_p = \dfrac{2d}{c}$.

Both observers must measure *c* for the speed of light. If *O* sees a greater distance traveled, then it must take longer for light to travel this distance in frame *O*,

$$\left(\frac{c\,\Delta t}{2}\right)^2 = \left(\frac{v\,\Delta t}{2}\right)^2 + d^2$$

Solving for the time interval in frame *O*, we find

$$\Delta t = \frac{2d/c}{\sqrt{1 - v^2/c^2}} = \frac{\Delta t_p}{\sqrt{1 - v^2/c^2}}$$

A rocket is moving at ¼ the speed of light relative to Earth. At the center of this rocket, a light suddenly flashes. To an observer at rest on Earth

[✓ A.] the light will reach the front of the rocket after it reaches the back of the rocket.

B. the light will reach the front of the rocket before it reaches the back of the rocket.

C. the light will reach the front of the rocket at the same instant that it reaches the back of the rocket.

A rocket is moving at ¼ the speed of light relative to Earth. At the center of this rocket, a light suddenly flashes. To an observer at rest in the rocket

A. the light will reach the front of the rocket after it reaches the back of the rocket.

B. the light will reach the front of the rocket before it reaches the back of the rocket.

[✓ C. the light will reach the front of the rocket at the same instant that it reaches the back of the rocket.]

An astronaut in an inertial reference frame measures a time interval Δt between her heartbeats. What will observers in all other inertial reference frames measure for the time interval between her heartbeats?

A. Less than Δt

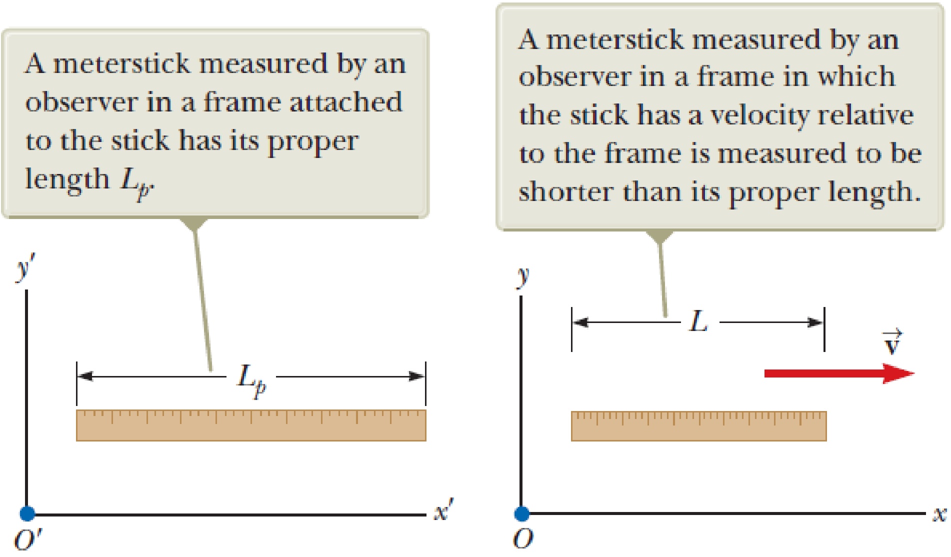
[✓ B. More than Δt]

C. Δt

D. The answer depends on whether they are moving toward her or away from her.

Relativity of lengths

The proper length L_p of an object is the length measured by an observer at rest relative to the object. The length of an object measured by someone in a reference frame that is moving with respect to the object is always less than the proper length. This effect is known as length contraction.



A meterstick measured by an observer in a frame attached to the stick has its proper length L_p.

A meterstick measured by an observer in a frame in which the stick has a velocity relative to the frame is measured to be shorter than its proper length.

Because of time dilation, the proper time interval is related to the *O* time interval by

$$\Delta t = \frac{\Delta t_p}{\sqrt{1 - v^2/c^2}}$$

Because the left end of the ruler moves to the right end of the ruler in the time Δt_p, *O′* concludes that the distance *L* between of the ruler is

$$L = v\,\Delta t_p = v\,\Delta t\,\sqrt{1 - v^2/c^2}$$

Therefore, we may write

$$L = L_p\sqrt{1 - v^2/c^2}$$

You are a passenger on a spaceship. As the speed of the spaceship increases, you would observe that

A. the length of your spaceship is getting shorter.

B. the length of your spaceship is getting longer.

[✓ C. the length of your spaceship is not changing.]

Compact equation using γ

We can rewrite our equations for time dilation and length contraction by defining

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

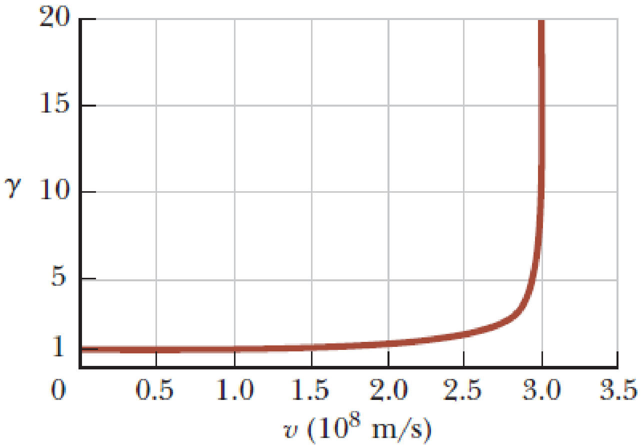
The equation for the dilated time interval with respect to the proper time interval becomes

$$\Delta t = \gamma\, \Delta t_p$$

The equation for length contraction with respect to the proper length becomes

$$L = L_p/\gamma$$

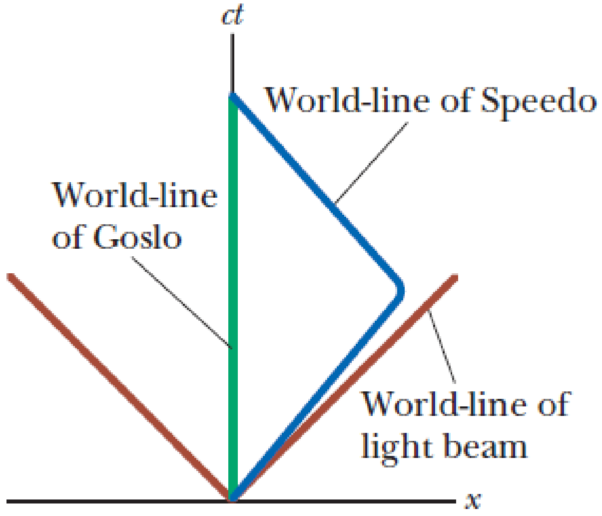
The γ parameter begins at unity and only slightly increases until the speed reaches about 90% of the speed of light. When the speed approach light speed, the γ parameter asymptotically increases.



The period of a pendulum is measured to be 3.00 s in the reference frame of the pendulum. What is the period when measured by an observer moving at a speed of 0.960c relative to the pendulum?

Identical twins Speedo and Goslo join a migration from the Earth to Planet X, 19.0 ly away in a reference frame in which both planets are at rest. The twins, of the same age, depart at the same moment on different spacecraft. Speedo's spacecraft travels steadily at 0.95 *c* and Goslo's at 0.60 *c*.

A path through space–time is called a world-line. For one-dimensional travel, the space-time graph has the *ct* variable as the ordinate and position *x* as the abscissa. We can use space-time graphs to help visualize problems, where it is clear that Speedo's time is more dilated than Goslo's time.



How much older is Goslo than Speedo after reaching planet X? (c = 1 ly/yr)

Suppose a runner moving at 0.75c carries a horizontal pole 15 m long toward a barn that is 10 m long. The barn has front and rear doors that are initially open. The barn doors can simultaneously close and open.

When the runner and the pole are inside the barn, the ground observer closes and then opens both doors so that the runner and pole are momentarily captured inside the barn and then proceed to exit the barn from the back doorway.

Do both the runner and the ground observer agree that the runner makes it safely through the barn?

Barn frame

$$L_{\mathrm{pole}} = L_p\sqrt{1-\frac{v^2}{c^2}} \approx 9.9\,\mathrm{m}$$

$$L_{\mathrm{barn}} = 10\,\mathrm{m}$$

Safe!

Pole frame

$$L_{\mathrm{pole}} = 10\,\mathrm{m}$$

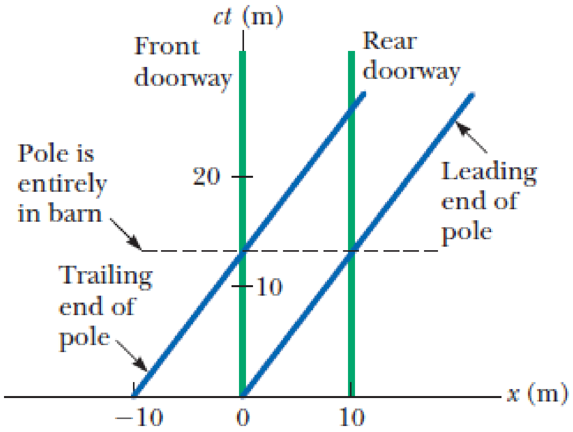
$$L_{\mathrm{barn}} = L_p\sqrt{1-\frac{v^2}{c^2}} \approx 6.6\,\mathrm{m}$$

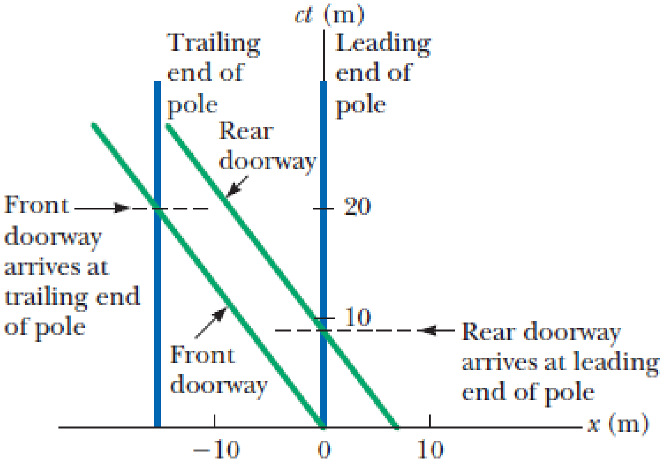
Not safe!

Uh oh!!!

The resolution of the “paradox” lies in the relativity of simultaneity. The closing of the two doors is measured to be simultaneous by the ground observer. Because the doors are at different positions, however, they do not close simultaneously as measured by the runner.

The rear door closes and then opens first, allowing the pole’s leading end to exit. The barn’s front door does not close until the pole’s trailing end passes.





The world-lines for the pole are separated by 15 m and vertical (pole at rest in the runner’s frame). The barn is hurtling toward the runner, so the world-lines for the front and rear doorways of the barn are tilted to the left. The world-lines for the barn are separated by 6.6 m, the contracted length as seen by the runner. The leading end of the pole leaves the rear doorway of the barn before the trailing end of the pole enters the barn. Thus, the opening of the rear door occurs before the closing of the front door.

Lorentz coordinate transformation

When an event occurs at point (*x*,*y*,*z*) at time *t* observed in frame *S*, what are the coordinates (*x*′,*y*′,*z*′) and time *t*′ of the event observed in frame *S*′ moving relative to *S* with constant speed *u* in the +*x*-direction?

The coordinate *x*′ is a proper length in *S*′, so in *S* it is contracted by the factor 1/γ. Therefore, instead of the classical Galilean transformation, we have

$$x = ut + x'/\gamma$$

Solving for *x*′ gives $\boxed{x' = \gamma\left(x - ut\right)}$

Because the frames only travel with different speeds in the *x*-direction, the other two coordinate transformations are simply

$$\boxed{y' = y} \qquad \boxed{z' = z}$$

The transformation from *S* to *S*′ is identical to that from *S*′ to *S*. Thus,

$$x' = -ut' + x/\gamma$$

Substituting this equation into the first boxed equation and solving for *t*′ gives

$$\boxed{t' = \gamma\left(t - ux/c^2\right)}$$

Lorentz velocity transformation

Using the *x*′ and *t*′ equations from the previous slide,

$$\begin{cases} x' = \gamma\left(x - ut\right) \\ t' = \gamma\left(t - ux/c^2\right) \end{cases}$$

We may write the differential,

$$\begin{cases} dx' = \gamma\left(dx - u\,dt\right) \\ dt' = \gamma\left(dt - \dfrac{u\,dx}{c^2}\right) \end{cases}$$

By definition, $v_x' = dx'/dt'$ and $v_x = dx/dt$. Dividing the equation for *dx*′ by *dt*′ gives

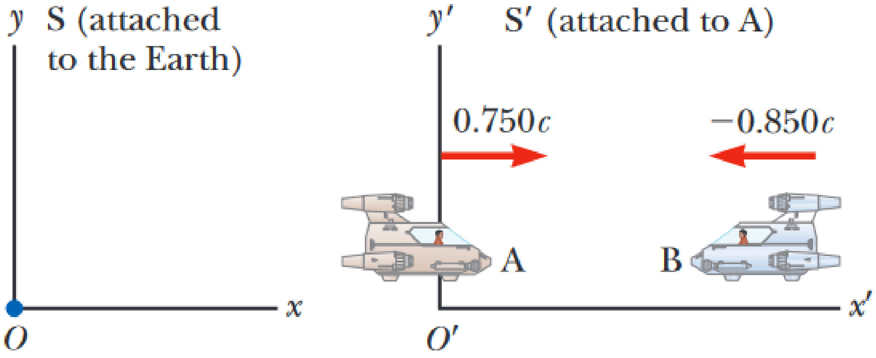
$$\boxed{v_x' = \frac{v_x - u}{1 - uv_x/c^2}}$$

Likewise, we may do the same transformation from *S*′ to *S* to get

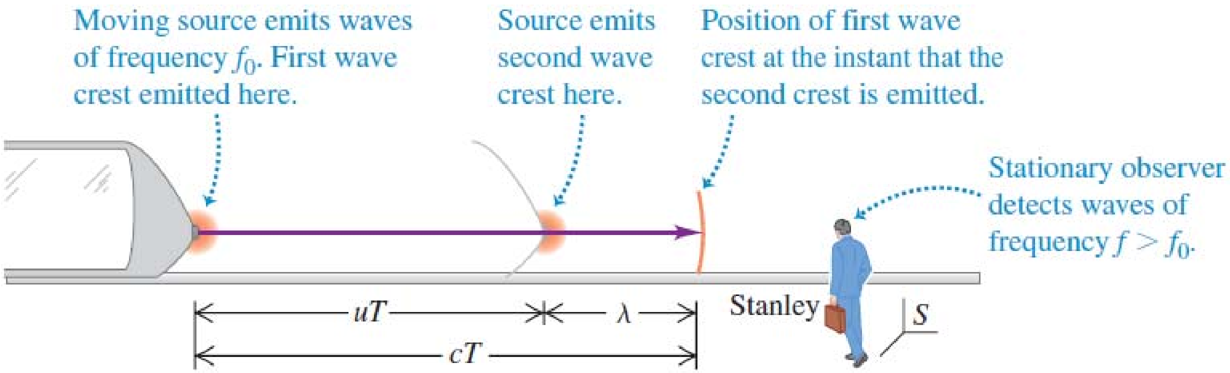
$$\boxed{v_x = \frac{v_x' + u}{1 + uv_x'/c^2}}$$

An extraterrestrial spaceship is moving away from the earth after an unpleasant encounter with its inhabitants. As it departs, the spaceship fires a missile toward the earth. An observer on earth measures that the spaceship is moving away with a speed of 0.600*c*. An observer in the spaceship measures that the missile is moving away from him at a speed of 0.800*c*. As measured by an observer on earth, how fast is the missile approaching the earth?

Two spacecraft *A* and *B* are moving in opposite directions as shown in the figure. An observer on the Earth measures the speed of spacecraft *A* to be 0.750*c* and the speed of spacecraft *B* to be 0.850*c*. Find the velocity of spacecraft *B* as observed by the crew on spacecraft *A*.



Doppler effect for electromagnetic waves



A source emits light with a wavelength of λ_0 at the speed of light c. The period of oscillation is T_0. The period of the light's oscillation by the observer in frame S is T.

The wavelength measured in the S frame is $\lambda = (c-u)T$ and the frequency is

$$f = \frac{c}{(c-u)T}$$

Due to time dilation, we know that $T = \gamma T_0$, which knowing $f_0 = c/T_0$, we may also write as

$$T = \frac{c}{f_0\sqrt{c^2-u^2}}$$

Substituting this equation into the top equation gives $\boxed{f = f_0\sqrt{\frac{c+u}{c-u}}}$

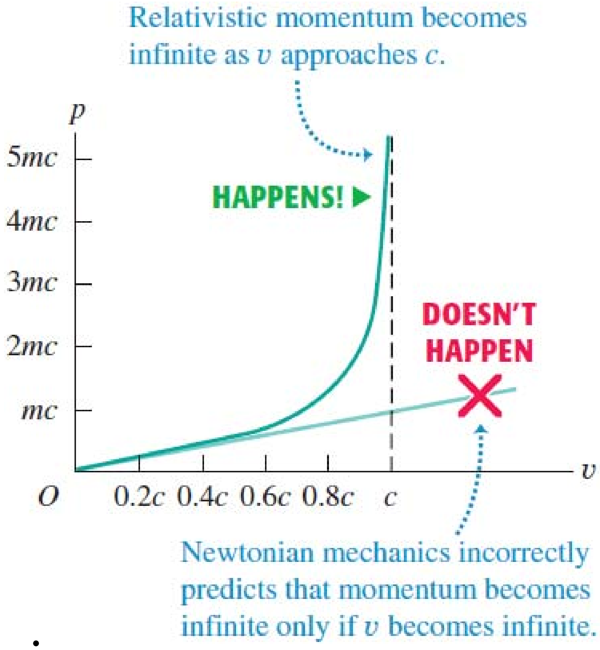
Relativistic momentum, mass, and acceleration

The relativistic momentum is a product of the velocity, γ parameter, and the rest mass,

$$\boxed{\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}}$$

Instead of using the rest mass, we could instead write the momentum as $\vec{p} = m_{\text{rel}}\vec{v}$, where the relativistic mass is

$$\boxed{m_{\text{rel}} = \frac{m}{\sqrt{1-v^2/c^2}}}$$


Relativistic momentum becomes infinite as v approaches c.

Newtonian mechanics incorrectly predicts that momentum becomes infinite only if v becomes infinite.

The relativistic generalization of Newton's second law is

$$\vec{F} = \frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

For one-dimensional motion, the equation can be written as a scalar, where

$$F = \frac{d}{dt}\frac{mv}{\sqrt{1-v^2/c^2}} = \frac{ma}{(1-v^2/c^2)^{3/2}}$$

The relativistic acceleration follows as $\boxed{a = \frac{F}{m}\left(1-\frac{v^2}{c^2}\right)^{3/2}}$

Momentum, mass, and acceleration with γ

Using the g parameter, we can compact the equations in the last slide to read

$$\vec{p} = \gamma m \vec{v}$$

$$m_{\mathrm{rel}} = \gamma m$$

A force in the direction of the object’s velocity has the relationship

$$F = \gamma^3 m a$$

When the force is perpendicular to the direction of the object’s velocity, such as in centripetal acceleration, the relationship is

$$F = \gamma m a$$

Relativistic work and kinetic energy

The rest energy *E* of an object is given by the famous equation $E = mc^2$.

The total energy of a freely moving relativistic object is

$$E + K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2$$

Thus, the kinetic energy of the object is $K = (\gamma - 1)\, mc^2$.

We may show this is the kinetic energy by observing the work done on the object by accelerating it from rest, where

$$W = \int_{x_1}^{x_2} F\, dx = \int_{x_1}^{x_2} \frac{ma}{(1 - v^2/c^2)^{3/2}}\, dx$$

Making the substitution, $a\, dx = \frac{dv}{dt} dx = dv \frac{dx}{dt} = v\, dv$, we may write

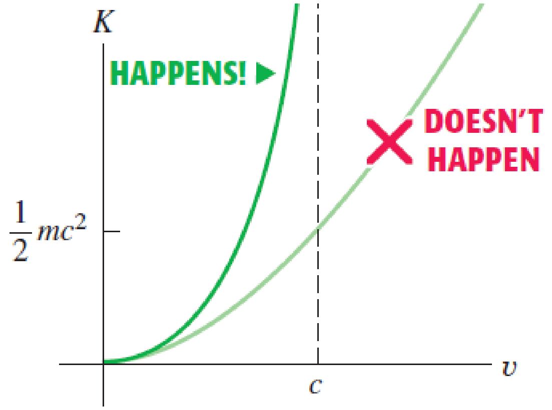
$$W = \int_0^v \frac{mv\, dv}{(1 - v^2/c^2)^{3/2}} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = (\gamma - 1)\, mc^2$$

Making use of the work-kinetic energy theorem $W = \Delta K$, this equates to the kinetic energy of the object.

Relativistic energy (a closer look)

The result for the relativistic kinetic energy, $K = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) mc^2$, can be expanded because $v^2/c^2 < 1$. This gives

$$K = \left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) - 1\right] mc^2$$



At velocities much smaller than *c*, the equation reduces to the classical kinetic energy

$$K \approx \frac{1}{2}mv^2$$

We may also rewrite the previous energy and momentum equations as

$$\left(\frac{E}{mc^2}\right)^2 = \frac{1}{1-v^2/c^2} \quad \text{and} \quad \left(\frac{p}{mc}\right)^2 = \frac{v^2/c^2}{1-v^2/c^2}$$

Subtracting the latter equation from the former equation, and then rearranging and taking the square root, results in the very useful expression

$$\boxed{E = \sqrt{p^2c^2 + m^2c^4}}$$

The special theory of relativity predicts that there is an upper limit to the speed of a particle. Therefore, there is an upper limit on which the following properties of a particle?

A. The kinetic energy
B. The linear momentum
C. The total energy
[✓ D. None of these]

State whether the following comments are **true** or **false**?

1. The rest mass of a proton depends on the observer's frame of reference.

True [✓ **False**]

2. The length of a meter stick depends on the observer's frame of reference.

[✓ **True**] **False**

3. The half-life of a muon depends on the observer's frame of reference.

[✓ **True**] **False**

4. The passage of time depends on the observer's frame of reference.

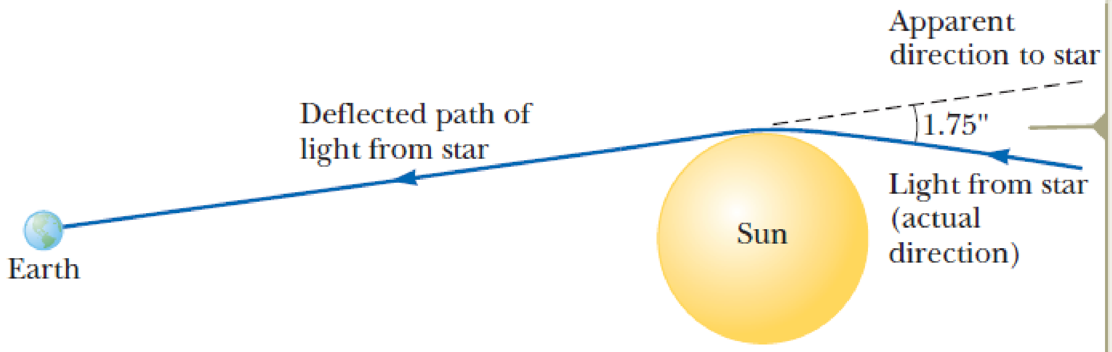
[✓ **True**] **False**

An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of 0.750 c. Find the magnitude of its relativistic momentum and compare this value with the momentum calculated from the classical expression.

Comment about general relativity

If we cannot distinguish experimentally between a uniform gravitational field at a particular location and a uniformly accelerated reference frame, then there cannot be any real distinction between the two.

Thus, a more general theory of relativity is necessary to also include such concepts as the observed gravitational redshifts and gravitational lensing of light (light has momentum and energy and is therefore affected by gravity).



In his general theory of relativity, Einstein calculated that starlight just grazing the Sun's surface should be deflected by an angle of 1.75 s of arc.