Exercise 2: darts, statistics, and the normal distribution

Purpose: investigate the probability distribution function for a measured quantity resulting from small random errors. Explore the difference between precision and accuracy.

Introduction

The statistical methods presented in earlier exercises are based on measured values of a quantity being symmetrically distributed about the true value. The spread in the values was assumed to be due to small random errors. In this case the limiting distribution describing the results was a bell-shaped curve centered on the true value. If properly normalized (the area beneath the curve equal to unity), the curve describes a probability distribution function for a given result. The most probable result is where the curve has a maximum. Improbable results are where the curve has small values. The probability for measuring a value x of a quantity with a true value of X is given by the *normalized Gauss distribution function* (specific probability distribution also known as the *normal distribution*),

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2} \ .$$

The factor σ characterizes the width of the curve. A broad curve has large σ corresponding to large uncertainty and poor precision of the measurements. A narrow curve has small σ corresponding to small uncertainty and good precision of the measurements. Verify the given Gauss distribution is correctly normalized by integration over all possible values of x. Present your calculation.

Laboratory assignment

Each group should grab a set of darts. The object is to hit the center target as many times as possible. Each dart represents a measurement of the quantity. A hit in the center target represents a measurement equal to the true value.

1. Take a few practice throws each.
2. Each student should throw darts at the center target (bin number 7). The total number of darts thrown per group should total no less than 200.
3. After each student has completed throwing the required number of darts at the target, count and tabulate the number of dart holes in each bin of the paper target. The bins are numbered 1 through 13. Report your data as a table similar to Table I.
4. Use these numbers to calculate the a) average bin hit, b) the standard deviation, and c) the standard error of the mean. Note that the units will be in “bins” and will unlikely be integers.
5. Groups compare their results to determine which group “measured” the quantity with greatest accuracy (average closest to the true value of 7) and which group “measured” the quantity with the greatest precision (smallest standard deviation).
6. Combine the data from each group to form the class data and place in another table similar to Table I.

Table I: darts.

Bin	# of darts
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

7. Using the combined data from the entire class, calculate the a) total number of darts thrown, b) average bin, c) the standard deviation, and d) the standard error of the mean for the class data.
8. Comment on any effects observed in your analysis from an increased number of "measurements."
9. Calculate the expected number of measurements of each value (dart holes in each bin) using the probability distribution function given in the introduction (see appendix for full details).
10. Make a table of the observed and expected number (not to be confused with the expectation value of the bin position) of darts in each bin for the entire class's results.
11. Calculate and report the *chi-squared* value for the observed number of darts compared with the expected number for the entire class's results. Use $Q_i^2 = \langle N_i \rangle_{\text{observed}}$, which is appropriate when comparing an integer number of observed measurements with an expected number. In this case the uncertainty Q_i is given by *Poisson statistics*. Also calculate and report the *reduced chi-squared* value. If a value of $\tilde{\chi}^2$ is order of 1 or less, then the expected distribution is supported. If a value of $\tilde{\chi}^2$ is much larger than 1, the expected distribution is not supported. State whether or not the normal distribution accurately describes the experimental results.

Equipment list: Darts, 8.5"x13" paper targets with 13-equal area bins.

Appendix I: formula for analyzing finite binned data representing a Gaussian distribution

For bins of equal width, the mean can be calculated using a simple average,

$$X = \frac{1}{N_{\text{darts}}} \sum_{i=1}^{N_{\text{bin}}} x_i n_i \,,$$

where x_i is the ith bin, n_i is the number of darts thrown in that bin, N_{bin} is the total number of bins, and N_{darts} is the total number of darts.

The standard deviation of binned data can be calculated by taking the square root of the variance. The standard deviation of the weighted sum follows as

$$\sigma = \frac{1}{\sqrt{N_{\text{darts}}}} \sqrt{\sum_{i=1}^{N_{\text{bin}}} (x_i - X)^2 n_i} \,.$$

Another important quantity is the standard error of the mean, which is an estimate of how far the sample mean is likely to be from the population mean. The standard error of the mean for a finite number of samples may be taken as

$$\text{SEM} = \frac{\sigma}{\sqrt{N_{\text{darts}}}} \,.$$

The expected number of darts at each bin location (estimated from the center of each bin) is found by simply multiplying the total number of darts thrown by the probability distribution defined as the variable N_{darts} multiplied by the probability distribution $P(x)$. For the normal distribution,

$$
\begin{aligned}
\langle n_i \rangle &= \int_{x_{i-1/2}}^{x_{i+1/2}} N_{\text{darts}} P(x)\, dx \\
&= \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{N_{\text{darts}}}{\sigma\sqrt{2\pi}}\, e^{-(x-X)^2/2\sigma^2}\, dx \\
&\approx \frac{N_{\text{darts}}}{\sigma\sqrt{2\pi}}\, e^{-(x_i-X)^2/2\sigma^2}\, \Delta x
\end{aligned}
$$

where you should use $\Delta x = 1\ \text{bin}$. Note that it is much simpler to use this approximation when tabulating your results in procedure 9.

The *chi-squared* value calculated from our tabulated data for the measured value of the number of darts thrown in each bin is given by

$$\chi^2 = \sum_{i=1}^{N_{\text{bin}}} \frac{(\langle n_i \rangle - n_i)^2}{Q_i^2}\,.$$

For Poisson statistics, the denominator Q_i^2 is replaced via the substitution $Q_i = \sqrt{n_i}$. This substitution is relevant for this laboratory and should be used.

The *reduced chi-squared* value is given by

$$\tilde{\chi}^2 = \frac{\chi^2}{d}\,,$$

where d is the number of degrees of freedom. The number of degrees of freedom d is the number of bins minus the number of constraints. Since the expected number of measurements is derived from three values characterizing the class data (total number of "measurements," average measured value, and standard deviation), the number of constraints is three. Thus, the number of degrees of freedom is 13 – 3 = 10.